

# State constrained patchy feedback stabilization

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## Abstract

We construct a patchy feedback for a general control system on  $\mathbb{R}^n$  which realizes practical stabilization to a target set  $\Sigma$ , when the dynamics is constrained to a given set of states  $S$ . The main result is that  $S$ -constrained asymptotically controllability to  $\Sigma$  implies the existence of a discontinuous practically stabilizing feedback. Such a feedback can be constructed in “patchy” form, a particular class of piecewise constant controls which ensure the existence of local Carathéodory solutions to any Cauchy problem of the control system and which enjoy good robustness properties with respect to both measurement errors and external disturbances.

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*Key Words*: asymptotic controllability, stabilization, state constraint, patchy feedback, robustness.

## 1 Introduction

Consider a general control system

$$\dot{x} = f(x, u) \quad x \in \mathbb{R}^d, \quad (1)$$

where the upper dot denotes a derivative w.r.t. time,  $u$  is the control taking values in a compact set  $\mathbf{U} \subset \mathbb{R}^m$  and  $f: \mathbb{R}^d \times \mathbf{U} \rightarrow \mathbb{R}^d$  is a vector field satisfying the following properties

**(F1)**  $f$  is continuous on  $\mathbb{R}^d \times \mathbf{U}$  and Lipschitz continuous in the variable  $x$ , uniformly for  $u \in \mathbf{U}$ , i.e. there exists a constant  $L_f$  such that

$$|f(x, u) - f(y, u)| \leq L_f |x - y|,$$

for all  $(x, u)$  and  $(y, u)$  in  $\mathbb{R}^d \times \mathbf{U}$ .

**(F2)**  $f$  has sub-linear growth, i.e.

$$|f(x, u)| \leq C_f (1 + |x|) \quad \forall x \in \mathbb{R}^d, \quad (2)$$

for some constant  $C_f$  independent on  $u$ .

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(F3) The set of velocities

$$f(x, \mathbf{U}) \doteq \{f(x, u) ; u \in \mathbf{U}\}$$

is convex for every  $x \in \mathbb{R}^d$ .

In this article, we want to address the problem of stabilizing trajectories of (1) towards a target set  $\Sigma \subseteq \mathbb{R}^d$  in the case of a dynamics constrained inside a prescribed set  $S \subset \mathbb{R}^d$ . In particular, we aim to construct a feedback control  $U = U(x)$  which realizes stabilization to a neighborhood of  $\Sigma$  and which is robust enough to provide the same stabilization also in presence of inner and outer perturbations of the dynamics, such as measurement errors and external disturbances.

However, one has to be careful because, even in very simple problems without state constraints, one cannot expect the existence of continuous control feedback laws which steer all trajectories towards a target  $\Sigma$  and stabilize them [19, 18, 8]. The lack of continuity in the feedback control creates quite a big theoretical problem, because continuity of  $U(x)$  is a minimal requirement to apply the classical existence theory of ordinary differential equations to the resulting closed loop system

$$\dot{x} = f(x, U(x)). \quad (3)$$

Therefore, in cases where discontinuous feedback laws have to be used, one has either to choose a generalized concept of solution or to verify that classical solutions still exist when a certain discontinuous law is used.

In order to precisely state our results, we need to first introduce a few definitions and notations. Namely, we denote with  $|x|$  the Euclidean norm of any element  $x \in \mathbb{R}^d$  and with

$$B_d \doteq \{x \in \mathbb{R}^d ; |x| < 1\}$$

the open unit ball of  $\mathbb{R}^d$ . Also, given any set  $E \subseteq \mathbb{R}^d$ , we denote its convex hull with  $\text{co}(E)$ , and its (topological) closure, interior and boundary respectively with  $\overline{E}$ ,  $\overset{\circ}{E}$  and  $\partial E$ , so that e.g. we have  $\overset{\circ}{B}_d = B_d$ ,  $\partial B_d = \{x \in \mathbb{R}^d ; |x| = 1\}$  and  $\overline{B}_d = B_d \cup \partial B_d$ . Finally, we use the notation  $v \bullet \phi$  for the directional derivative of a function  $\phi$  along the direction  $v$ , i.e.

$$v \bullet \phi(x) \doteq \lim_{t \rightarrow 0} \frac{\phi(x + tv) - \phi(x)}{t}. \quad (4)$$

Given a feedback control  $u(x)$ , we recall that for a system of differential equations like (1) with initial datum  $x(0) = x_o$ , a Carathéodory solution on some interval  $I$  is an absolutely continuous map  $t \mapsto x(t)$  which satisfies (1) for a.e.  $t \in I$ , i.e. satisfying the integral representation

$$x(t) = x_0 + \int_0^t f(x(s), u(x(s))) ds \quad \forall t \in I. \quad (5)$$

**Definition 1.1** *Given a bounded constraint set  $S \subset \mathbb{R}^d$  and a target set  $\Sigma \subset \mathbb{R}^d$  such that  $S \cap \Sigma \neq \emptyset$ , we say that the system (1) is open loop  $S$ -constrained controllable to  $\Sigma$  if the following holds. For any initial state  $x_o \in S$ , there exists a Lebesgue measurable control function  $u(\cdot)$  and a time  $T = T(x_o, u) \geq 0$  such that denoting with  $x(\cdot)$  the Carathéodory solution, corresponding to the control  $u$ , of the Cauchy problem for (1) with initial datum  $x(0) = x_o$ , one has*

$$x(t) \in S \quad \forall t \in [0, T],$$

and

$$x(T) \in \Sigma.$$

**Definition 1.2** Given a bounded constraint set  $S \subset \mathbb{R}^d$  and a target set  $\Sigma \subset \mathbb{R}^d$  such that  $S \cap \Sigma \neq \emptyset$ , we say that a feedback control  $U: \text{dom } U \rightarrow \mathbf{U}$ , defined on some open domain  $\text{dom } U$  which contains  $S \setminus \Sigma$ , is  $S$ -constrained stabilizing to  $\Sigma$  in Carathéodory sense for the system (1) if the following holds. For any initial state  $x_o \in S$ , the closed loop system (3) with initial datum  $x(0) = x_o$  admits Carathéodory solutions and, moreover, there exists  $T = T(x_o) \geq 0$  such that for any Carathéodory solution  $x(\cdot)$  to (3) starting from  $x_o$  one has

$$x(t) \in S \quad \forall t \in [0, T_{\max}[ ,$$

and

$$x(t) \in \Sigma \quad \forall t \in [T, T_{\max}[ ,$$

where we denoted by  $T_{\max} = T_{\max}(x_o, U)$  the maximal time of existence of  $x(\cdot)$ .

One has to be careful when dealing with Definition 1.2. Indeed, as we have already stressed, in general there might fail to exist a continuous feedback law  $U(x)$  which stabilizes (1). Hence, one has to consider discontinuous feedback controls, but in such a case there might be no Carathéodory solutions at all.

To cope with this problem, we choose here to consider a particular class of feedback controls, the so called *patchy feedbacks* [1, 2, 3, 7], which are piecewise constant and such that the resulting control system (3) always admits local Carathéodory solutions for positive times.

We remark that this is not the only possible approach to deal with discontinuous control laws. Indeed, one could allow for arbitrary discontinuous feedback controls  $u = u(x)$  and replace Carathéodory trajectories with a weaker concept of solutions. In recent years many authors have followed this alternative path by considering sample-and-hold solutions and Euler solutions for discontinuous vector fields (see e.g. [10]) and several results have also been obtained in the context of constrained dynamics (see [11, 12, 13]). However, with this approach, in order to guarantee that the resulting control is robust with respect to both inner and outer perturbations, without chattering phenomena, it is necessary to impose additional assumptions on the sampling step of the solutions considered. On the contrary, patchy feedbacks allow to consider classical Carathéodory solutions and do not require any additional hypothesis to be robust, thanks to their regularity which ensures that only “tame” discontinuities are present in the dynamics.

Let us introduce now the assumptions on the constraint set  $S$ . First we recall a notion from non-smooth analysis [10]: given a closed set  $S$  in  $\mathbb{R}^d$  and a point  $x \in S$ , the *Clarke (proximal) normal cone* to  $S$  in  $x$  is defined as the set

$$N_S^C(x) \doteq \left\{ \lambda \xi ; \lambda \geq 0, \xi \in \overline{\text{co}} \left( \{0\} \cup \left\{ v = \lim_{v_i \rightarrow 0} \frac{v_i}{|v_i|} ; v_i \perp S \text{ in } x_i, x_i \rightarrow x \right\} \right) \right\}, \quad (6)$$

where “ $v_i \perp S$  in  $x_i$ ” means that  $v_i + x_i \notin S$  and  $x_i$  belongs to the projection of  $v_i + x_i$  on  $S$ , or equivalently

$$v_i + x_i \notin S \quad \text{and} \quad |v_i| = \inf_{\xi \in S} |v_i + x_i - \xi|.$$

We are now ready to state the main hypotheses on  $S$ :

**(S1)**  $S$  is compact and wedged at each  $x \in \partial S$ . The latter means that at each boundary point  $x$  one has that  $N_S^C(x)$ , the Clarke normal cone to  $S$  in  $x$ , is pointed; that is,

$$N_S^C(x) \cap \{-N_S^C(x)\} = \{0\}.$$

**(S2)** The following “strict inwardness” condition holds:

$$\min_{u \in \mathbf{U}} f(x, u) \cdot p < 0,$$

for all  $x \in \partial S$  and  $p \in N_S^C(x) \setminus \{0\}$ .

Notice that, in Definition 1.1, we do not assume *stability* of the target set, i.e. we are not requiring that trajectories starting sufficiently close to  $\Sigma$  always remain close to  $\Sigma$ . Hence, in general we do not expect a feedback which stabilizes the dynamics precisely to the target. The main result of this paper concerns instead *practical stabilization* of (1) to  $\Sigma$ , i.e. the existence for all  $\delta > 0$  of a patchy feedback control which stabilizes trajectories of (1) to a neighborhood  $\Sigma^\delta \doteq \Sigma + \delta B_d$  of the target set.

**Theorem 1** *Assume that the system (1) satisfies (F1)–(F3) and open loop  $S$ -constrained controllability to  $\Sigma$ , where  $S$  is a set satisfying (S1) and (S2) and  $\Sigma$  is any closed set such that  $S \cap \Sigma \neq \emptyset$ . Then, for every  $\delta > 0$  there exists a patchy feedback control  $U = U(x)$ , defined on an open domain  $\mathcal{D}$  with  $S \setminus \Sigma^\delta \subseteq \mathcal{D}$ , which is  $S$ -constrained stabilizing to  $\Sigma^\delta$  for (1).*

Under assumptions (S1) and (S2) on the constraint set, it was proved in [12] that it is possible to construct a discontinuous feedback control which steers Euler solutions of (1) to  $\Sigma^\delta$ . However, as mentioned above, when a patchy feedback exists more robustness properties can be expected to hold than in the case of the generic feedback presented in [12]. This indeed happens also for the constrained problem considered here, and we will show that practical stabilization of perturbed systems can be established as well. We refer to Section 6.3 for the result about perturbed systems and for further discussions about robustness of the control provided by Theorem 1.

It is worth to mention that in the process of constructing the required feedback controls, we also prove two technical Lemmas 3.1 and 3.2, whose applications extend beyond the specific usage in this paper. They offer, indeed, general procedures to construct patchy feedbacks starting from semiconcave functions and wedged sets, respectively, and hence represent powerful mathematical tools on their own. An example of application of Lemma 3.1 is presented in Section 6.4, where we give a new proof of a result by Ancona and Bressan [1] about the existence of a stabilizing patchy feedback for systems with unconstrained dynamics (1) which are GAC to the origin.

## 2 Preliminaries

### 2.1 Patchy vector fields and patchy feedbacks

We start by recalling the main definitions and properties of the class of discontinuous vector fields (*patchy vector fields*) introduced in [1].

**Definition 2.1** *We say that  $g : \Omega \rightarrow \mathbb{R}^d$  is a patchy vector field on the open domain  $\Omega \subseteq \mathbb{R}^d$  if there exists a family  $\{(\Omega_\alpha, g_\alpha) ; \alpha \in \mathcal{A}\}$  such that (see Fig. 1)*

- (i)  $\mathcal{A}$  is a totally ordered set of indices;
- (ii) each  $\Omega_\alpha$  is an open domain with smooth boundary;
- (iii) the open sets  $\Omega_\alpha$  form a locally finite covering of  $\Omega$ ;
- (iv) each  $g_\alpha$  is a Lipschitz continuous vector field defined on a neighborhood of  $\overline{\Omega}_\alpha$ , which points strictly inward at each boundary point  $x \in \partial\Omega_\alpha$ : namely, calling  $\mathbf{n}(x)$  the outer normal at the boundary point  $x$ , we require

$$g_\alpha(x) \cdot \mathbf{n}(x) < 0 \quad \forall x \in \partial\Omega_\alpha ; \quad (7)$$

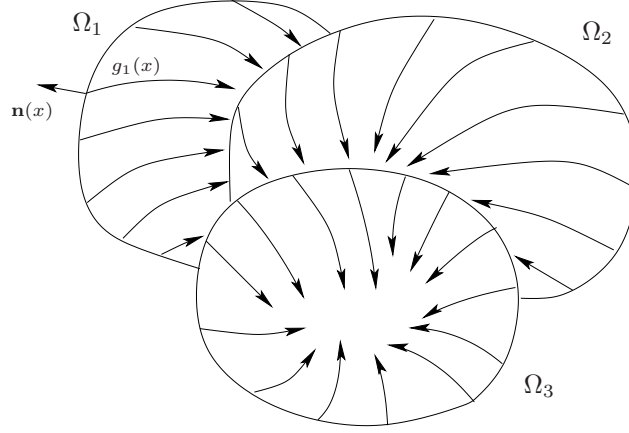


Figure 1: A patchy vector field.

(v) the vector field  $g$  can be written in the form

$$g(x) = g_\alpha(x) \quad \text{if} \quad x \in \Omega_\alpha \setminus \bigcup_{\beta > \alpha} \Omega_\beta. \quad (8)$$

Each element  $(\Omega_\alpha, g_\alpha)$  of the family is called patch.

By defining

$$\alpha^*(x) \doteq \max \{ \alpha \in \mathcal{A} ; x \in \Omega_\alpha \}, \quad (9)$$

the identity (8) can be written in the equivalent form

$$g(x) = g_{\alpha^*(x)}(x) \quad \forall x \in \Omega. \quad (10)$$

We shall occasionally adopt the longer notation  $(\Omega, g, (\Omega_\alpha, g_\alpha)_{\alpha \in \mathcal{A}})$  to indicate a patchy vector field, specifying both the domain and the single patches.

**Remark 2.1** Notice that the smoothness assumption on the boundaries  $\partial\Omega_\alpha$  in (ii) above can be relaxed. Indeed, one can consider patches  $(\Omega_\alpha, g_\alpha)$  where the domain  $\Omega_\alpha$  only has piecewise smooth boundary. In this case, the inward-pointing condition (7) can be rephrased as

$$g(x) \in \overset{\circ}{T}_\Omega(x), \quad (11)$$

$T_\Omega(x)$  denoting the (Bouligand) tangent cone to  $\Omega$  at the point  $x$ , defined by (see [10])

$$T_\Omega(x) \doteq \left\{ v \in \mathbb{R}^d ; \liminf_{t \downarrow 0} \frac{d(x + tv, \Omega)}{t} = 0 \right\}. \quad (12)$$

Clearly, at any regular point  $x \in \partial\Omega$ , the interior of the tangent cone  $T_\Omega(x)$  is precisely the set of all vectors  $v \in \mathbb{R}^d$  that satisfy  $v \cdot \mathbf{n}(x) < 0$  and hence (11) coincides with the inward-pointing condition (7).

**Remark 2.2** Notice also that in Definition 2.1 the values attained by  $g_\alpha$  on  $\Omega_\alpha \cap \Omega_\beta$  for any  $\beta > \alpha$  are irrelevant. Similarly, the inward pointing condition (7) does not really matter in points  $x \in \partial\Omega_\alpha \cap \Omega_\beta$ , for any  $\beta > \alpha$ , and in points  $x \in \partial\Omega_\alpha \cap (\mathbb{R}^d \setminus \Omega)$ . This is a consequence of

the fact that, in general, the patches  $(\Omega_\alpha, g_\alpha)$  are not uniquely determined by the patchy vector field  $g$ .

Indeed, as observed in [1], whenever a Lipschitz vector field  $h_\alpha$  is given on  $\overline{\Omega}_\alpha$  so that it verifies (7) on  $(\partial\Omega_\alpha \cap \Omega) \setminus \bigcup_{\beta>\alpha} \Omega_\beta$ , one can always construct another Lipschitz vector field  $g_\alpha$  on  $\overline{\Omega}_\alpha$  such that  $g_\alpha = h_\alpha$  on  $(\overline{\Omega}_\alpha \cap \Omega) \setminus \bigcup_{\beta>\alpha} \Omega_\beta$  and such that (7) is verified at every  $x \in \partial\Omega_\alpha$ .

If  $g$  is a patchy vector field, the differential equation

$$\dot{x} = g(x) \quad (13)$$

has many interesting properties. In particular, it was proved in [1] that, given any initial condition

$$x(0) = x_0, \quad (14)$$

the Cauchy problem (13)–(14) has at least one forward solution, and at most one backward solution in Carathéodory sense. We recall that a Carathéodory solution of (13)–(14) on some interval  $I$  is an absolutely continuous map  $t \mapsto \gamma(t)$  which satisfies (13) for a.e.  $t \in I$ , i.e.

$$\gamma(t) = x_0 + \int_0^t g(\gamma(s)) ds \quad \forall t \in I. \quad (15)$$

We collect below the other main properties satisfied by trajectories of (13)–(14).

- For every Carathéodory solution  $\gamma(\cdot)$  of (13), the map  $t \mapsto \alpha^*(\gamma(t))$ , with  $\alpha^*$  the function defined in (9), is left continuous and non-decreasing. Moreover, it is piecewise constant on every compact interval  $[a, b]$ , i.e. there exist a partition  $a = t_0 < t_1 < \dots < t_N = b$  of  $[a, b]$  and indices  $\alpha_1 < \dots < \alpha_N$  in  $\mathcal{A}$  such that  $\alpha^*(\gamma(t)) = \alpha_i$  for all  $t \in ]t_{i-1}, t_i]$ .
- The set of all Carathéodory solutions of (13)–(14) is closed in the topology of uniform convergence, but possibly not connected.
- Carathéodory solution of (13) are robust w.r.t. to both inner and outer perturbations; namely, for any solution  $y(\cdot)$  of the perturbed system

$$\dot{y} = g(y + \zeta) + d,$$

there exists a solution  $x(\cdot)$  of the unperturbed system (1) such that  $\|x - y\|_{\mathbf{L}^\infty}$  is as small as we want, provided that  $\zeta$  and  $d$  are small enough in  $\mathbf{BV}$  and  $\mathbf{L}^1$ , respectively (see [2] for the details in the general case, and Section 6 for a discussion of the constrained case).

The class of patchy vector fields is of great interest in a wide variety of control problems for general nonlinear control systems (1), that can be solved by constructing a state feedback  $u = U(x)$  which renders the resulting closed loop map  $g(x) = f(x, U(x))$  a patchy vector field and, hence, ensures robustness properties of the resulting solutions without additional efforts. This leads to the following definition.

**Definition 2.2** Let  $\mathcal{A}$  be a totally ordered set of indices, and  $(U_\alpha)_{\alpha \in \mathcal{A}}$  be a family of control values in  $\mathbf{U}$  such that, for each  $\alpha \in \mathcal{A}$ , there exists a patch  $(\Omega_\alpha, g_\alpha)$  which satisfies

$$g_\alpha(x) = f(x, U_\alpha) \quad \forall x \in \Omega_\alpha \setminus \bigcup_{\beta>\alpha} \Omega_\beta. \quad (16)$$

If the family  $\{\Omega_\alpha\}_{\alpha \in \mathcal{A}}$  form a locally finite covering of the set  $\bigcup_{\alpha \in \mathcal{A}} \Omega_\alpha$ , then the piecewise constant map

$$U(x) \doteq U_\alpha \quad \text{if} \quad x \in \Omega_\alpha \setminus \bigcup_{\beta>\alpha} \Omega_\beta \quad (17)$$

is called a patchy feedback control on  $\bigcup_{\alpha \in \mathcal{A}} \Omega_\alpha$ .

By requiring (16) with  $(\Omega_\alpha, g_\alpha)$  being a patch, in particular we require that

$$f(x, U_\alpha(x)) \cdot \mathbf{n}(x) < 0 \quad \forall x \in \partial\Omega_\alpha \setminus \bigcup_{\beta > \alpha} \Omega_\beta.$$

By Definitions 2.1–2.2 it is thus clear that, given a patchy feedback  $U$ , the corresponding collection of patches  $(\Omega_\alpha, g_\alpha)$ ,  $\alpha \in \mathcal{A}$ , defines a patchy vector field  $g(x) = f(x, U(x))$  on  $\bigcup_{\alpha \in \mathcal{A}} \Omega_\alpha$ . Moreover, recalling the definition of  $\alpha^*(x)$  in (9), a patchy feedback control can be written in the equivalent form

$$U(x) = U_{\alpha^*(x)}(x) \quad x \in \Omega \doteq \bigcup_{\alpha \in \mathcal{A}} \Omega_\alpha. \quad (18)$$

We shall occasionally adopt the longer notation  $(U, (\Omega_\alpha, U_\alpha)_{\alpha \in \mathcal{A}})$  to indicate a patchy feedback control, similarly to the notation adopted for patchy vector fields.

**Remark 2.3** *As in Remark 2.2, the values attained by  $U_\alpha$  on the set  $\Omega_\alpha \cap \Omega_\beta$  are irrelevant, whenever  $\alpha < \beta$ , and similarly it only matters that  $f(\cdot, U_\alpha(\cdot))$  fulfills the inward-pointing condition (7) at points of  $(\partial\Omega_\alpha \cap \Omega) \setminus \bigcup_{\beta > \alpha} \Omega_\beta$ .*

Finally, we recall the definition of the lexicographic order on a set of  $k$ -tuple indices. Given  $\sigma = (\sigma_1, \dots, \sigma_k)$  and  $\sigma' = (\sigma'_1, \dots, \sigma'_k)$ , we say that  $\sigma \prec \sigma'$  iff

$$\exists j \in \{1, \dots, k\} \quad \text{such that} \quad \begin{cases} \sigma_i = \sigma'_i & \text{for all } i < j, \\ \sigma_j < \sigma'_j. \end{cases} \quad (19)$$

## 2.2 Wedged sets and inner approximations

In this section we collect a few geometrical properties which are satisfied by either any wedged set or specifically by sets for which both **(S1)** and **(S2)** hold. As a first step, we want to give a characterization of wedged sets.

We start by recalling the definition of Clarke's tangent cone to  $S$  in  $x$ , which we denote by  $T_S^C(x)$ , as the polar cone to the normal cone  $N_S^C(x)$  defined in (6), that is

$$T_S^C(x) \doteq \{v \in \mathbb{R}^d ; p \cdot v \leq 0 \quad \forall p \in N_S^C(x)\}. \quad (20)$$

In general, the Clarke tangent cone is smaller than the Bouligand tangent cone defined in (12) (see [10]), i.e. there holds for every closed set  $S$  and every  $z \in S$

$$T_S^C(z) \subseteq T_S(z).$$

For later use, we recall that whenever  $A, B \subseteq \mathbb{R}^d$  with  $A$  wedged,  $z \in A \cap B$  and  $\overset{\circ}{T}_A^C(z) \cap T_B^C(z) \neq \emptyset$ , one can prove (see [16]) that

$$T_A^C(z) \cap T_B^C(z) \subseteq T_{A \cap B}^C(z) \subseteq T_{A \cap B}(z). \quad (21)$$

Moreover, for all  $v \in \mathbb{R}^d$  and  $\varepsilon > 0$  we call *wedge* of axis  $v$  and radius  $\varepsilon$  the set (see Figure 2 left)

$$\mathcal{W}(v, \varepsilon) \doteq \{sw ; w \in v + \varepsilon B_d, s \in [0, \varepsilon]\}.$$

Finally, to denote the “lower” part of the boundary of a wedge (see Figure 2 right), we use the following

$$\partial^- \mathcal{W}(v, \varepsilon) \doteq \{\varepsilon w ; w \in v + \varepsilon \partial B_d, (v - w) \cdot v \leq 0\}.$$

We are now in a position to state the following characterization result.

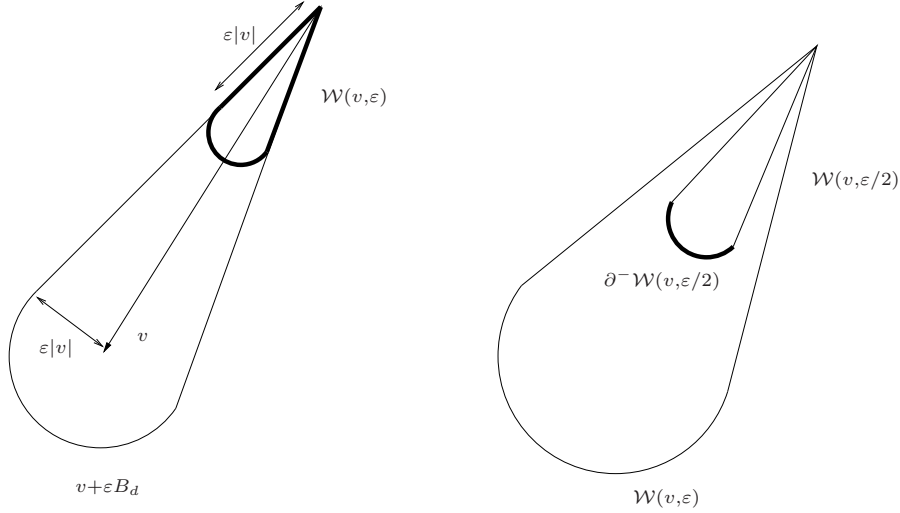


Figure 2: Left: Wedge  $\mathcal{W}(v, \varepsilon)$ , of axis  $v$  and radius  $\varepsilon$ . Right: The “lower” boundary  $\partial^- \mathcal{W}(v, \varepsilon/2)$  of the smaller wedge is strictly separated from  $\mathbb{R}^d \setminus \mathcal{W}(v, \varepsilon)$ .

**Proposition 2.1** *Let  $S \subseteq \mathbb{R}^d$  be a closed nonempty set and  $x \in \partial S$ . Then, the following properties are equivalent:*

- (i)  $S$  is wedged in  $x$ ;
- (ii)  $T_S^C(x)$  has nonempty interior;
- (iii) there exist  $v \in \mathbb{R}^d$  and  $\varepsilon > 0$  such that

$$y + \mathcal{W}(v, \varepsilon) \subset S \quad \forall y \in \{x + \varepsilon B_d\} \cap S.$$

We refer to [10] for the proof of the equivalences above. We also mention that wedged sets are sometimes called *epi-Lipschitz sets* because they are locally the epigraph of a Lipschitz continuous function (see [15]).

For later use, we need a better understanding of the behavior of wedges when their radii are rescaled. It is immediate to deduce from the definition that  $\mathcal{W}(v, \varepsilon/2) \subseteq \mathcal{W}(v, \varepsilon)$  for every  $v \in \mathbb{R}^d$  and  $\varepsilon > 0$ . Moreover, we claim that points of the “lower” boundary  $z \in \partial^- \mathcal{W}(v, \varepsilon/2)$ , are well inside the larger wedge  $\mathcal{W}(v, \varepsilon)$  (see again Figure 2 right). Indeed, it is not difficult to verify that for any fixed  $v \in \mathbb{R}^d$  and any  $z \in \partial^- \mathcal{W}(v, \varepsilon/2)$ , there holds

$$\frac{\varepsilon}{\mathcal{O}(1)} \leq d(z, \overline{\mathbb{R}^d \setminus \mathcal{W}(v, \varepsilon)}) \leq \varepsilon \mathcal{O}(1). \quad (22)$$

Next, we collect a few results concerning the “inner approximations” of a set  $S$  satisfying (S1) and (S2). Given a closed set  $S$  and  $r \geq 0$ , we call  $r$ -inner approximation of  $S$  the set

$$S_r \doteq \{x \in \mathbb{R}^d ; d(x, \mathbb{R}^d \setminus S) \geq r\}, \quad (23)$$

and we define

$$Q(S, r) \doteq S \setminus \overset{\circ}{S}_r. \quad (24)$$

Given  $x \in S$ , we also set  $r(x) \doteq d(x, \overline{\mathbb{R}^d \setminus S})$ .



**Lemma 2.1 (Lemma 3.3 in [11])** *Let  $S$  be a set such that (S1) is verified. Then there exists  $r_o > 0$  such that for all  $r \in [0, r_o]$  the set  $S_r$  is nonempty and wedged in every point of its boundary.*

Observe that in terms of Clarke's tangent cone (20) to a closed set  $S$ , condition (S2) can be restated as follows. For every  $x \in \partial S$  there holds

$$T_S^C(x) \cap f(x, \mathbf{U}) \neq \emptyset,$$

i.e. (S2) ensures that there is an admissible speed pointing strictly inside the set  $S$ . In fact, next lemma shows that, by combining properties (S1) and (S2), it is possible to prove a stronger property.

**Lemma 2.2 (Lemma 3.5 in [11])** *Assume that  $f$  in (1) satisfies (F1)–(F3). Let  $S$  be a set such that (S1) and (S2) are verified and  $r_o > 0$  be the value found in Lemma 2.1. Then there exist  $\mu > 0$  and a Lipschitz continuous function  $v: Q(S, r_o) \rightarrow \mathbb{R}^d$  such that*

$$v(x) \in f(x, \mathbf{U}) \quad \forall x \in Q(S, r_o) \quad (25)$$

and

$$v(x) + \mu B_d \subset T_{S_{r(x)}}^C(x) \quad \forall x \in Q(S, r_o). \quad (26)$$

Notice that Lemma 2.2 makes explicit use of the convexity hypothesis (F3) on the vector field, in order to obtain (25). Also, one can deduce from wedgedness of  $S$  and (26) that  $v(x) \neq 0$  for all  $x \in \mathbb{R}^n$ .

**Lemma 2.3 (Lemma 3.6 in [11])** *Assume that  $f$  in (1) satisfies (F1)–(F3). Let  $S$  be a set such that (S1) and (S2) are verified,  $r_o > 0$  be the value found in Lemma 2.1 and  $v: Q(S, r_o) \rightarrow \mathbb{R}^d$  the Lipschitz continuous function found in Lemma 2.2. Then there exists  $\tilde{\varepsilon} > 0$  such that for every  $x \in Q(S, r_o)$  one has*

$$\begin{aligned} y + \mathcal{W}(v(x), \tilde{\varepsilon}) &\subset S_{r(x)} & \forall y \in \{x + \tilde{\varepsilon} B_d\} \cap S_{r(x)}, \\ y + \mathcal{W}(-v(x), \tilde{\varepsilon}) &\subset \overline{\mathbb{R}^d \setminus S_{r(x)}} & \forall y \in \{x + \tilde{\varepsilon} B_d\} \setminus \overset{\circ}{S}_{r(x)}. \end{aligned}$$

Lemma 2.3 provides uniformity of the radii of wedges with axis  $v(x)$  which are contained in each inner approximation  $S_{r(x)}$ . Moreover, it implies that  $v(x)$  is in the interior of the Clarke tangent cone  $T_{S_{r(x)}}^C(x)$  for all  $x \in Q(S, r_o)$ . While this result is not completely surprising, it shows all its importance when combined with the following one, dealing with decrease properties of the signed distance function. We recall that, given a closed set  $Z \subset \mathbb{R}^d$ , the *signed distance* of a point  $x \in \mathbb{R}^d$  from  $Z$  is given by

$$\Delta_Z(x) \doteq d(x, Z) - d(x, \overline{\mathbb{R}^d \setminus Z}).$$

It is not difficult to verify that the function  $\Delta_Z$  is Lipschitz continuous.

**Lemma 2.4 (Lemma 3.7 in [11])** *Let  $S$  be a closed set which is wedged at  $x \in \partial S$ . Assume  $v \in \mathbb{R}^d$  and  $\varepsilon > 0$  are such that*

$$\begin{aligned} y + \mathcal{W}(v, \varepsilon) &\subset S & \forall y \in \{x + \varepsilon B_d\} \cap S, \\ y + \mathcal{W}(-v, \varepsilon) &\subset \overline{\mathbb{R}^d \setminus S} & \forall y \in \{x + \varepsilon B_d\} \setminus \overset{\circ}{S}. \end{aligned}$$

*Then there exists a neighborhood  $\mathcal{N}_x$  of  $x$  such that*

$$\nabla \Delta_S(y) \cdot v \leq -\varepsilon,$$

*for all  $y \in \mathcal{N}_x \setminus \partial S$  in which  $\Delta_S$  is differentiable.*

### 2.3 Strong CLF families

In this section we recall the definition of strong CLF family related to a control problem with constraint  $S$  and target  $\Sigma$ , as introduced in [13], and its main properties. As before, we use in this section the notation

$$\Sigma^\delta \doteq \Sigma + \delta B_d,$$

for any  $\delta > 0$ .

**Definition 2.3** *A strong control Lyapunov functions family (or strong CLF family) w.r.t.  $S$  and  $\Sigma$  is a family of functions  $\{\varphi_\gamma(\cdot)\}$  for which there exist  $\varepsilon > 0$  and  $C > 0$  such that for every  $\gamma > 0$  the following properties hold*

- (i)  $\varphi_\gamma$  is Lipschitz continuous and locally semiconcave on  $S + \varepsilon B_d$ ;
- (ii) for every  $x \in \{S + \varepsilon B_d\} \setminus \overline{\Sigma^{2\gamma}}$  in which  $\varphi_\gamma$  is differentiable, one has

$$\min_{w \in \mathcal{F}} \nabla \varphi_\gamma(x) \cdot w \leq -C, \quad (27)$$

where  $\mathcal{F} = T_S^C(x) \cap f(x, \mathbf{U})$  if  $x \in \partial S$  and  $\mathcal{F} = f(x, \mathbf{U})$  if  $x \notin \partial S$ ;

- (iii)  $\varphi_\gamma > 0$  on  $S \setminus \overline{\Sigma^\gamma}$  and  $\varphi_\gamma \equiv 0$  in  $S \cap \overline{\Sigma^\gamma}$ .

The main result we need is the following proposition that can be deduced from the proof of Theorem 4.1 in [12] (see also the characterization theorem given in Theorem 2.3 of [13], in the case of finite time controllability).

**Proposition 2.2 (Clarke & Stern [12])** *Assume that  $f$  in (1) satisfies (F1)–(F3) and that  $S$  satisfies (S1) and (S2). If the system (1) satisfies open loop  $S$ –constrained controllability to  $\Sigma$  then the system (1) admits a strong CLF family w.r.t.  $S$  and  $\Sigma$ .*

*In particular, such a family  $\{\varphi_\gamma\}$  can be chosen so that the following holds. Let  $r_o > 0$  be the value found in Lemma 2.1,  $v: Q(S, r_o) \rightarrow \mathbb{R}^d$  be the Lipschitz continuous function found in Lemma 2.2 and  $C$  the constant in (27). Then, for all  $\gamma > 0$  there exists  $r_\gamma \in ]0, r_o]$  such that*

$$\nabla \varphi_\gamma(x) \cdot v(x) \leq -\frac{C}{2}, \quad (28)$$

*for all  $x \in Q(S, r_\gamma)$  in which  $\varphi_\gamma$  is differentiable.*

In fact, one can require inequalities (27) and (28) to hold also in points where  $\varphi_\gamma$  is not differentiable, by replacing  $\nabla \varphi_\gamma(x)$  with any vector in the limiting subgradient of  $\varphi_\gamma$  in  $x$  (see [10] for a precise definition of generalized gradients and their properties in the context of non-smooth analysis). However, in view of Lipschitz continuity of the functions  $\varphi_\gamma$  and of Rademacher's theorem, the set of points  $\mathcal{N}$  where  $\nabla \varphi_\gamma$  does not exist is Lebesgue negligible and, for the results we want to prove in this paper, it is enough to work outside  $\mathcal{N}$ .

## 3 Fundamental Lemmas

We collect in this section two lemmas which provide the key ingredients for the proof of Theorem 1. Namely, these results offer a general procedure to construct a patchy feedback whenever there exists a semiconcave function which decreases along trajectories of (1), or whenever we are close to the boundary of a set which satisfies (S1) and (S2).

It should be noted that the procedure outlined in Lemma 3.1 strictly follows the one exploited in [7] to construct a nearly optimal control for an unconstrained dynamics.

Anyway, the present formulation is more general since it makes no use of any specific features of the optimality problem and therefore it offers a flexible and effective tool to construct patchy feedbacks for general problems, well beyond the specific applications treated here. For completeness sake, the proof of this lemma is included in the Appendix.

**Lemma 3.1** *Assume that  $f$  in (1) satisfies **(F1)** and **(F2)**. Let  $\Omega \subseteq \mathbb{R}^d$  be a set,  $V: \Omega \rightarrow [0, +\infty[$  be a locally semiconcave function on  $\Omega$  and  $h: \mathbb{R}^d \rightarrow [0, +\infty[$  be a continuous function such that*

$$\min_{u \in \mathbf{U}} \{ \nabla V(x) \cdot f(x, u) \} + h(x) \leq 0, \quad (29)$$

*for all  $x \in \Omega$  where  $\nabla V(x)$  is defined. Then for every bounded set  $\Lambda \subset \subset \Omega$ , every  $\rho > 0$  and every  $0 < \varepsilon < \max_{\bar{\Lambda}} h(x)$ , there exist a continuous function  $W: \mathbb{R}^d \rightarrow [0, +\infty[$  and a patchy feedback control  $U: \mathcal{D} \rightarrow \mathbf{U}$ , with  $\mathcal{D} \supseteq \bar{\Lambda} \setminus \mathcal{E}$  and  $\mathcal{E} \supseteq \{x \in \Lambda; h(x) \leq \varepsilon\}$ , such that the following properties hold.*

(i)  *$W(x)$  is the pointwise minimum of a finite family of quadratic functions  $W_1, \dots, W_q: \mathbb{R}^d \rightarrow [0, +\infty[$  defined by*

$$W_i(x) = \kappa |x - x_i|^2 + r_i, \quad i = 1, \dots, q, \quad (30)$$

*for suitable  $x_i \in \mathbb{R}^d$ ,  $r_i \in \mathbb{R}$  and a common constant  $\kappa > 0$ , and  $W$  satisfies*

$$V(x) \leq W(x) \leq V(x) + \rho, \quad (31)$$

*for all  $x \in \Lambda$ .*

(ii) *For all  $x \in \mathcal{D} \setminus \mathcal{E}$  one has*

$$f(x, U(x)) \bullet W(x) + h(x) \leq \varepsilon. \quad (32)$$

*where the symbol “ $\bullet$ ” stands for the directional derivative, as in (4).*

Moreover, one can require that the following additional properties hold.

(iii) *For every  $\lambda > 0$ , the function  $W$  can be constructed so that, for every  $x \in \Lambda$  and every  $j \in \{1, \dots, q\}$  such that  $W(x) = W_j(x)$ , there exists  $y_j \in \bar{\Lambda}$  where  $V$  is differentiable and*

$$|x - y_j| \leq \lambda, \quad |\nabla V(y_j) - \nabla W_j(x)| \leq \lambda. \quad (33)$$

(iv) *For every  $\lambda' > 0$ , the patchy control  $U = (U, (\Omega_\alpha, U_\alpha)_{\alpha \in \mathcal{A}})$  can be constructed so that it satisfies  $\text{diam } \Omega_\alpha \leq \lambda' \ \forall \alpha \in \mathcal{A}$ .*

**Remark 3.1** *Notice that the assumptions on  $f$  in Lemma 3.1 can be actually relaxed to mere boundedness and uniform continuity on the set  $\bar{\Lambda} \times \mathbf{U}$ . Also, if  $h$  is bounded below by a positive constant  $c$ , then for  $\varepsilon < c$  the choice  $\mathcal{E} = \emptyset$  is allowed, i.e. for  $\varepsilon > 0$  small enough the corresponding patchy feedback  $U$  can be defined in the whole compact  $\bar{\Lambda}$ . Finally, any change of the parameter  $\lambda' > 0$  in (iv) only affects the patchy control, while the function  $W$  requires no modification.*

The result in Lemma 3.1 can be summarized as follows: given a semiconcave function  $V$  such that (29) is satisfied and a bounded set  $\Lambda$ , we can always provide a smoother approximation of  $V$  over  $\Lambda$ , in terms of a piecewise quadratic function  $W$  which makes  $|W - V|$  as small as we want, and we can construct a patchy feedback  $U$  such that  $W$  satisfies (32), which is an approximate version of (29). Additionally, the construction can be required to satisfy properties (iii) and (iv): namely, we can choose  $W$  so that if  $W = W_j$ , then  $\nabla W_j$  is close to  $\nabla V$  evaluated in a nearby point and the domains  $\Omega_\alpha$  which constitutes  $U$  can be taken as small as we want.

Now, we introduce the second result needed by Theorem 1, i.e. a lemma dealing with the construction of a patchy feedback control  $U(x)$  near the boundary of a wedged set  $S$  so that  $S$  results positively invariant for the resulting dynamics (3). Notice that the sets  $S_r$  and  $Q(S, r)$ , for any  $r > 0$ , have been introduced in (23)–(24) and that for any  $x \in S$  we use  $r(x)$  to denote the quantity  $d(x, \overline{\mathbb{R}^d \setminus S})$ , as in Section 2.

**Lemma 3.2** *Assume that  $f$  in (1) satisfies (F1)–(F3) and that  $S$  satisfies (S1) and (S2). Let  $r_o > 0$  be as in Lemma 2.1,  $v: Q(S, r_o) \rightarrow \mathbb{R}^d$  be as in Lemma 2.2 and  $\tilde{\varepsilon}$  be as in Lemma 2.3. Then, for all  $\tilde{r} \in ]0, r_o[$  and all  $\varepsilon \in ]0, \tilde{\varepsilon}/2[$ , there exists a patchy feedback control  $U: \mathcal{D} \rightarrow \mathbf{U}$ , with*

$$Q(S, \tilde{r}) \subseteq \mathcal{D} \subseteq Q(S, \tilde{r}) + (r_o - \tilde{r})\overline{B_d}, \quad (34)$$

such that for all  $x \in Q(S, \tilde{r})$  there hold

$$|f(x, U(x)) - v(x)| < \varepsilon, \quad (35)$$

$$y + \mathcal{W}(f(x, U(x)), \varepsilon) \subset S_{r(x)} \quad \forall y \in \{x + \tilde{\varepsilon}B_d\} \cap S_{r(x)}, \quad (36)$$

$$y + \mathcal{W}(-f(x, U(x)), \varepsilon) \subset \overline{\mathbb{R}^d \setminus S_{r(x)}} \quad \forall y \in \{x + \tilde{\varepsilon}B_d\} \setminus \overset{\circ}{S}_{r(x)}. \quad (37)$$

Moreover, one can require that for every  $\lambda > 0$  the patchy control  $U = (U, (\Omega_\alpha, U_\alpha)_{\alpha \in \mathcal{A}})$  satisfies  $\text{diam } \Omega_\alpha \leq \lambda$  and  $\Omega_\alpha \cap Q(S, r_o) \neq \emptyset$ ,  $\forall \alpha \in \mathcal{A}$ .

This lemma allows to construct a patchy control  $U$ , defined on the whole  $Q(S, \tilde{r})$ , such that wedges of axis  $f(x, U(x))$  and uniform radii are contained in each inner approximation  $S_{r(x)}$ . In particular, by applying Lemma 2.4 to the vector  $v = f(x, U(x))$ , to the wedged sets  $S_{r(x)}$  and to the point  $x \in \partial S_{r(x)}$ , conditions (36) and (37) imply that there exists a neighborhood  $\mathcal{N}_x$  of  $x$  such that

$$\nabla \Delta_{S_{r(x)}}(y) \cdot f(x, U(x)) \leq -\varepsilon < 0, \quad (38)$$

whenever  $y \in \mathcal{N}_x$  is a point of differentiability of the map  $\xi \mapsto \Delta_{S_{r(x)}}(\xi)$ .

In the applications to systems whose dynamics is constrained to a wedged set  $S$ , it should now seem quite natural to proceed as follows: we adopt the construction provided by Lemma 3.2 in the region  $Q(S, \tilde{r})$  which is near the boundary  $\partial S$ ; and we use a different construction, based on Lemma 3.1, to deal with the region  $S_{\tilde{r}}$ , which is the part of  $S$  sufficiently far from  $\partial S$ . However, Lemma 3.1 does not keep explicitly track of the location of trajectories, and hence the domain  $\mathcal{D}$  of the resulting control might well be larger than the constraint set  $S$ .

In order to actually piece together the dynamics given by the different feedbacks in  $Q(S, \tilde{r})$  and  $S_{\tilde{r}}$ , so that they can be combined into an  $S$ -constrained stabilizing patchy feedback defined in the whole set, the following will be useful.

**Remark 3.2** *In the same settings of Lemma 3.1, there exist a constant  $\sigma > 0$  and, for any compact sets  $K \subseteq \Lambda \setminus \mathcal{E}$ , a totally ordered set of indices  $\mathcal{B} \subseteq \mathcal{A}$  such that the following properties hold*

$$(i) \ K \subseteq \bigcup_{\beta \in \mathcal{B}} \Omega_\beta \subseteq K + \lambda' \overline{B_d};$$

$$(ii) \text{ setting } M^* \doteq \max_{\overline{\Lambda}} W, \ m^* \doteq \min_{\overline{\Lambda} \setminus \mathcal{E}} W, \ N \doteq \lfloor \frac{M^* - m^*}{\sigma} \rfloor + 1 \text{ and}$$

$$\mathcal{L}_m \doteq \{x \in \mathbb{R}^d; \ M^* - (m+1)\sigma \leq W(x) < M^* - m\sigma\} \quad m \in \{0, \dots, N-1\},$$

there exist indices  $\beta_o < \dots < \beta_N$  in  $\mathcal{B}$  such that  $\mathcal{B} = \{\beta; \beta_o \leq \beta \leq \beta_N\}$ ,

$$K \cap \overline{\mathcal{L}}_m \subseteq \bigcup_{\substack{\beta \in \mathcal{B} \\ \beta_m \leq \beta < \beta_{m+1}}} \Omega_\beta, \quad (39)$$

and for any fixed  $\beta \in \mathcal{B}$ , if  $\beta_m \leq \beta < \beta_{m+1}$  then one has

$$f(x, U_\beta(x)) \cdot \mathbf{n}(x) < 0 \quad \forall x \in \partial\Omega_\beta \setminus \left( \bigcup_{\beta' > \beta} \Omega_{\beta'} \cup \bigcup_{m' > m} \overline{\mathcal{L}}_{m'} \right). \quad (40)$$

In other words, given a compact set  $K \subseteq \Lambda \setminus \mathcal{E}$  we can extract, from the family of domains  $\{\Omega_\alpha\}_{\alpha \in \mathcal{A}}$  used to construct  $U$ , a subfamily  $\{\Omega_\beta\}_{\beta \in \mathcal{B}}$  which provides a covering of  $K$  contained in a  $\lambda'$ -neighborhood of  $K$ . Moreover, the resulting subfamily can be described in more detail by considering the sets  $K \cap \mathcal{L}_m$ , with  $m \in \{0, \dots, N-1\}$ , which are uniform slices of the set  $K$  w.r.t. sublevel sets of the function  $W$ . Namely, the family  $\{\Omega_\beta\}_{\beta \in \mathcal{B}}$  is ordered so that, for all  $m$ , the elements  $\{\Omega_\beta\}$  covering  $K \cap \mathcal{L}_m$  have smaller indices  $\beta$  than the elements  $\{\Omega_{\beta'}\}$  covering  $K \cap \mathcal{L}_{m+1}$ ; and the vector field corresponding to the “extracted” control  $\tilde{U} = (\tilde{U}, (\Omega_\beta, U_\beta)_{\beta \in \mathcal{B}})$  is inward pointing on each  $\partial\Omega_\beta$  as long as we haven’t entered a patch or a slice with larger index. It is worth noticing that such a procedure does not require any change to the original construction of  $W$  and  $U$ . Anyway, the choice of the subfamily  $(\Omega_\beta, U_\beta)_{\beta \in \mathcal{B}}$  instead of  $(\Omega_\alpha, U_\alpha)_{\alpha \in \mathcal{A}}$  has a drawback: in general the resulting control  $\tilde{U}$  is only patchy on the domain  $\bigcup_{\beta \in \mathcal{B}} \Omega_\beta$  and the inward pointing condition might fail in points of the set

$$\mathcal{O}_{\beta, m'} \doteq (\partial\Omega_\beta \cap \mathcal{L}_{m'}) \setminus \left( K \cup \bigcup_{\substack{\beta' \in \mathcal{B} \\ \beta < \beta' < \beta_{m+1}}} \Omega_{\beta'} \right),$$

for  $\beta_m \leq \beta < \beta_{m+1}$  and  $m' > m$ .

**Remark 3.3** In Lemma 3.1, the requirement of  $W$  being the pointwise minimum of a finite family of quadratic functions means that its level sets are contained in a finite union of spheres. As a result, also the boundaries of the slices  $\mathcal{L}_m$  in Remark 3.2, which are sets of the form

$$\{W = c_m\} \doteq \{x \in \mathbb{R}^d; \ W(x) = c_m\},$$

are contained in the union of a finite number of spheres. We now claim that, up to a slight modification of the slices  $\mathcal{L}_m$ , it is not restrictive to assume that all these spheres are pairwise non-tangent.

Indeed, whenever the level set  $\{W = \bar{c}\}$  is contained in the union of spheres where two or more are tangent, we can always consider a level set  $\{W = c'\}$ , with  $c'$  arbitrarily close to  $\bar{c}$ , that is contained in the finite union of spheres with no tangent intersections among them. Hence, since the domains  $\{\Omega_\beta\}_{\beta \in \mathcal{B}}$  in Remark 3.2 are open sets and because of the uniform continuity of  $W$  and  $f$  on the compact set  $\overline{\Lambda}$ , we can always replace the slices  $\mathcal{L}_0, \dots, \mathcal{L}_{N-1}$  with slightly different slices such that (39) and (40) are still verified and the boundaries are contained in unions of pairwise non-tangent spheres.

## 4 Proof of the main result

The plan of the proof is the following. First, in Step 1–Step 4, we combine the lemmas from Section 3 in order to construct a feedback control on  $S$ . Then, in Step 5–Step 7, we prove that such a feedback is patchy and realize the required  $S$ -constrained practical stabilization.

**Step 1.** Since we are assuming that the system is  $S$ -constrained globally asymptotically controllable, then by Proposition 2.2 there exists a strong CLF family w.r.t.  $S$  and  $\Sigma$ . Let us denote such a family by  $\{\varphi_\gamma\}$ .

Fixed any  $\delta > 0$ , we want to construct a patchy feedback  $U$  such that any trajectory starting from  $x_o \in S \setminus \Sigma^\delta$  tends to  $\Sigma^\delta$ , always remaining inside  $S$ . Let  $\bar{\gamma} > 0$  be any number such that  $4\bar{\gamma} \leq \delta$ ,  $\bar{\varphi} = \varphi_{\bar{\gamma}}$  be the corresponding semiconcave function in the strong CLF family and  $\bar{r} = r_{\bar{\gamma}} \in ]0, r_o]$  be the value given in Proposition 2.2 so that, for all  $x \in Q(S, \bar{r})$  where  $\bar{\varphi}$  is differentiable, one has

$$\nabla \bar{\varphi}(x) \cdot v(x) \leq -\frac{C}{2}, \quad (41)$$

$v$  being the Lipschitz continuous function from Lemma 2.2 and  $C$  being the constant from (27) in Definition 2.3.

**Step 2.** Now we apply Lemma 3.1 to the semiconcave function  $V = \bar{\varphi}$  and to the constant function  $h \equiv C$  over the set  $\Omega = \{S + \varepsilon B_d\} \setminus \overline{\Sigma^{2\bar{\gamma}}}$ , with  $\Lambda = S \setminus \Sigma^{3\bar{\gamma}}$  and  $\rho = \bar{r}/4$  and  $\varepsilon = C/4$ , so that we get a function  $W$  which is the pointwise minimum of a finite family of quadratic functions  $W_1, \dots, W_M$  of the form (30) and a patchy feedback  $U^\sharp = (U^\sharp, (P_\alpha, p_\alpha)_{\alpha \in \mathcal{A}'})$  on a domain  $\mathcal{D}^\sharp \supseteq \Lambda$  such that (31) and (32) are satisfied, i.e. such that

$$\bar{\varphi} \leq W \leq \bar{\varphi} + \frac{\bar{r}}{4}, \quad (42)$$

on  $\Lambda$  and

$$f(x, U^\sharp(x)) \bullet W(x) \leq -C + \frac{C}{4} < -\frac{C}{8}, \quad (43)$$

for all  $x \in \mathcal{D}^\sharp$ . Notice that, the particular structure form (30) of the functions  $W_j$  implies that, for every direction  $\mathbf{v}$  and every point  $x$ , there exists the directional derivative of  $\mathbf{v} \bullet W(x)$  and it actually coincides with  $\mathbf{v} \bullet W_k(x) = \nabla W_k(x) \cdot \mathbf{v}$  for one of the indices  $k$  such that  $W(x) = W_k(x)$ . Moreover, by denoting with  $L_v$  a Lipschitz constant for  $v$  and with  $L_W$  a Lipschitz constant for  $W$  on  $Q(S, \bar{r}/2)$  and by choosing

$$\Delta \doteq \max\{|v(x)| ; x \in Q(S, r_o)\} + L_v L_W, \quad \lambda \doteq \frac{\varepsilon}{\Delta},$$

for all  $x \in Q(S, \bar{r}/2) \cap \Lambda = Q(S, \bar{r}/2) \setminus \Sigma^{3\bar{\gamma}}$  and all  $j \in \{1, \dots, M\}$  such that  $W(x) = W_j(x)$ , by (iii) there exists  $y_j \in \overline{B(x, \lambda)} \cap \overline{\Lambda}$  such that  $\nabla \bar{\varphi}(y_j)$  exists and  $|\nabla W_j(x) - \nabla \bar{\varphi}(y_j)| \leq \lambda$ . In particular, this happens for the index  $\iota$  such that  $v(x) \bullet W(x) = \nabla W_\iota(x) \cdot v(x)$ . Summing up, we obtain

$$\begin{aligned} v(x) \bullet W(x) &= \nabla W_\iota(x) \cdot v(x) \leq \nabla \bar{\varphi}(y_\iota) \cdot v(y_\iota) + |v(y_\iota)| |\nabla W_\iota(x) - \nabla \bar{\varphi}(y_\iota)| + L_W |v(x) - v(y_\iota)| \\ &\leq \nabla \bar{\varphi}(y_\iota) \cdot v(y_\iota) + \lambda (|v(y_\iota)| + L_W L_v) \leq -\frac{C}{2} + \varepsilon = -\frac{C}{4}. \end{aligned} \quad (44)$$

Also, by choosing  $\lambda' = \rho = \bar{r}/4$  it is not restrictive to assume  $\text{diam } P_\alpha \leq \bar{r}/4$  for all  $\alpha \in \mathcal{A}'$ . Finally, following Remark 3.2 we can find  $\sigma > 0$  and, for  $K = S_{\bar{r}/2} \setminus \Sigma^{3\bar{\gamma}}$ , a family of indices  $\mathcal{B} \subseteq \mathcal{A}'$  such that properties (i) and (ii) of the remark are verified for the piecewise constant

feedback  $\tilde{U}^\sharp = (\tilde{U}^\sharp, (P_\beta, p_\beta)_{\beta \in \mathcal{B}})$ . In particular,

$$K \subseteq \bigcup_{\beta \in \mathcal{B}} P_\beta \subseteq K + \frac{\bar{r}}{4} \overline{B_d} \subseteq S_{\bar{r}/4} \quad (45)$$

**Step 3.** Next, we notice that, since  $W$  is Lipschitz continuous on  $Q(S, \bar{r}/2)$ , (44) implies the existence of  $\bar{\varepsilon} > 0$ , depending on  $\bar{\gamma}$ , such that

$$|\omega - v(x)| < \bar{\varepsilon} \implies \omega \bullet W(x) < -\frac{C}{8}, \quad (46)$$

for all  $x \in Q(S, \bar{r}/2) \setminus \Sigma^{3\bar{\gamma}}$ , and of course it is not restrictive to assume that  $\bar{\varepsilon} < \tilde{\varepsilon}/2$ , where  $\tilde{\varepsilon}$  is the value from Lemma 2.3.

Hence, we apply Lemma 3.2 to  $\bar{r}/2$  and  $\bar{\varepsilon}$  to obtain a patchy feedback control  $U^b = (U^b, (Q_\alpha, q_\alpha)_{\alpha \in \mathcal{A}})$  defined on a domain  $\mathcal{D}^b \supseteq Q(S, \bar{r}/2)$  such that (35), (36) and (37) are satisfied for  $x \in Q(S, \bar{r}/2)$ . In particular, by combining (35) with (46), we get the inequality

$$f(x, U^b(x)) \bullet W(x) < -\frac{C}{8}, \quad (47)$$

for all  $x \in Q(S, \bar{r}/2) \setminus \Sigma^{3\bar{\gamma}}$ . Also, it is not restrictive to assume that  $\text{diam } Q_\alpha \leq \lambda$  for a suitable constant  $\lambda > 0$  which will be chosen at the end of Step 5.

**Step 4.** Now we modify and relabel the pairs of domains and controls

$$\{(Q_\alpha, q_\alpha) ; \alpha \in \mathcal{A}\}, \quad \{(P_\beta, p_\beta) ; \beta \in \mathcal{B}\},$$

in such a way that the resulting family gives a patchy feedback on the whole  $S$ . As in Remark 3.2, we define the sets

$$\mathcal{L}_m \doteq \{x \in \mathbb{R}^d ; M^* - (m+1)\sigma \leq W(x) < M^* - m\sigma\} \quad m \in \{0, \dots, N-1\}$$

where  $M^* \doteq \max_{\bar{\Lambda}} W$ ,  $m^* \doteq \min_{\bar{\Lambda}} W$  and  $N \doteq \lfloor \frac{M^* - m^*}{\sigma} \rfloor + 1$ , the set  $\Lambda$  still being  $S \setminus \Sigma^{3\bar{\gamma}}$  as in Step 2. We also define

$$\mathcal{L}_N \doteq \{x \in \mathbb{R}^d ; W(x) < M^* - N\sigma\}.$$

For  $j = 0, \dots, N$  set

$$\mathcal{A}_j \doteq \{\alpha \in \mathcal{A} ; Q_\alpha \cap \bar{\mathcal{L}}_j \neq \emptyset\}, \quad \Gamma_{j,0,\alpha} \doteq Q_\alpha \setminus \left( \bigcup_{i < j} \mathcal{L}_i \right) \quad \forall \alpha \in \mathcal{A}_j,$$

and for  $j = 0, \dots, N-1$  set

$$\mathcal{B}_j \doteq \{\beta \in \mathcal{B} ; \beta_j \leq \beta < \beta_{j+1}\}, \quad \Gamma_{j,1,\beta} \doteq P_\beta \quad \forall \beta \in \mathcal{B}_j,$$

being  $\beta_0 < \dots < \beta_N$  the indices from Remark 3.2. Observe that, by construction, one has

$$\bar{\mathcal{L}}_j \cap Q(S, \bar{r}/2) \subseteq \bigcup_{\alpha \in \mathcal{A}_j} \Gamma_{j,0,\alpha}, \quad j = 0, \dots, N, \quad (48)$$

$$\bar{\mathcal{L}}_j \cap (S_{\bar{r}/2} \setminus \Sigma^{3\bar{\gamma}}) \subseteq \bigcup_{\beta \in \mathcal{B}_j} \Gamma_{j,1,\beta}, \quad j = 0, \dots, N-1. \quad (49)$$

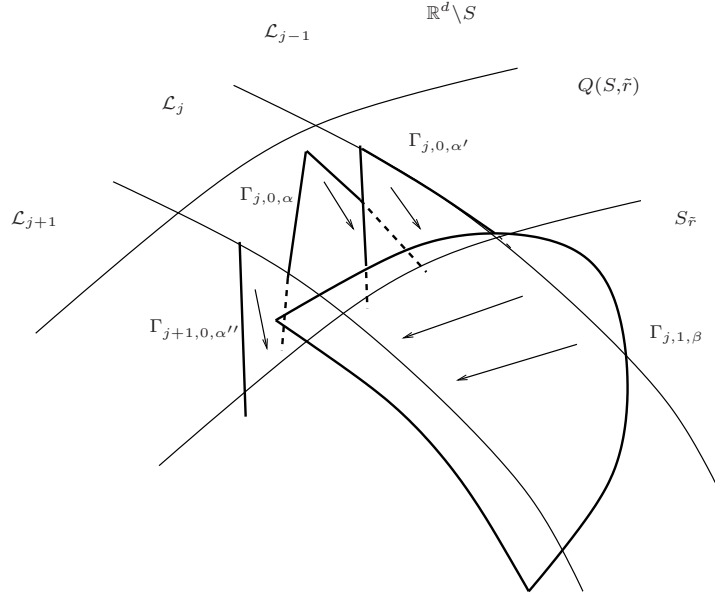


Figure 3: The covering  $\{\Gamma_c\}_{c \in \mathcal{C}}$  close to  $\partial S_{\bar{r}}$ , for indices  $\alpha < \alpha' < \alpha'' \in \mathcal{A}$  and  $\beta \in \mathcal{B}$ . Notice that  $x(\cdot)$  might start from  $\Gamma_{j,0,\alpha}$ , then cross  $\Gamma_{j,0,\alpha'}$  and  $\Gamma_{j,1,\beta}$  and finally reach  $\Gamma_{j+1,0,\alpha''}$ .

Hence,  $\{\Gamma_c\}_{c \in \mathcal{C}}$  is a locally finite covering of  $S \setminus \Sigma^{3\bar{\gamma}}$ , with a set of indices

$$\mathcal{C} \subseteq \{0, \dots, N\} \times \{0, 1\} \times (\mathcal{A} \cup \mathcal{B}),$$

which is totally ordered through the lexicographic order (see also Figure 3). Moreover, on each  $\Gamma_c$  a constant control is defined by

$$u_c \doteq u_{j,i,\eta} = \begin{cases} q_\eta & \text{if } i = 0, \\ p_\eta & \text{if } i = 1. \end{cases}$$

We now claim that, in fact:

- $U = (U, (\Gamma_c, u_c)_{c \in \mathcal{C}})$  is a patchy feedback control on the domain

$$\mathcal{D} \doteq \left( \mathcal{D}^b \cup \bigcup_{\beta \in \mathcal{B}} P_\beta \right) \setminus \Sigma^{3\bar{\gamma}} \supseteq (K \cup Q(S, \bar{r}/2)) \setminus \Sigma^{3\bar{\gamma}} \supseteq S \setminus \Sigma^\delta;$$

- the corresponding trajectories of (1) with initial datum  $x_o \in S$  never exit the set  $S$ ;
- $U$  stabilizes the trajectories of (1) to  $\Sigma_\delta$ .

In the next steps we prove this claim, completing the proof.

**Step 5.** To prove that  $U$  is a patchy feedback control on  $\mathcal{D} \supseteq S \setminus \Sigma^\delta$ , we have to verify that for all  $z \in \partial \Gamma_c \cap \overset{\circ}{\mathcal{D}}$  in which the inward pointing condition fails, one has  $z \in \Gamma_{c'}$  for some  $c' \in \mathcal{C}$  with  $c' > c$ .



**a.** Consider first the case of  $z \in \partial\Gamma_c \cap \overset{\circ}{\mathcal{D}} \cap S = \partial\Gamma_c \cap (S \setminus \Sigma^{3\bar{\gamma}})$ . Let us assume that  $c = (j, 1, \beta)$  for some  $j = 0, \dots, N-1$  and  $\beta \in \mathcal{B}_j$ . In this case, by definition  $\Gamma_c = P_\beta$  and then if

$$z \in \partial P_\beta \setminus \left( \bigcup_{\beta' > \beta} P_{\beta'} \cup \bigcup_{j' > j} \mathcal{L}_{j'} \right),$$

the inward pointing condition in  $z$  is guaranteed by (40). Assume now that  $z \in \partial P_\beta \cap P_{\beta'}$  for some  $\beta' > \beta$ . Then, either  $\beta < \beta' < \beta_{j+1}$  or  $\beta' \in \mathcal{B}_{j'}$  for some  $j' > j$ . In the former case  $z \in \Gamma_{j,1,\beta'}$  and in the latter case  $z \in \Gamma_{j',1,\beta'}$  and we always conclude  $z \in \Gamma_{c'}$  for some  $c' > c$ . Finally, assume that for some  $j' > j$  there holds

$$z \in (\partial P_\beta \cap \mathcal{L}_{j'}) \setminus \left( \bigcup_{\beta' > \beta} P_{\beta'} \right). \quad (50)$$

In this case,  $z$  cannot belong to  $K = S_{\bar{r}/2} \setminus \Sigma^{3\bar{\gamma}}$ , because otherwise (39) would imply

$$z \in \mathcal{L}_{j'} \cap K \subseteq \bigcup_{\beta_{j'} \leq k < \beta_{j'+1}} P_k \subseteq \bigcup_{k > \beta} P_k,$$

which contradicts (50). Thus,  $z \in S \setminus S_{\bar{r}/2}$  and hence  $z \in \mathcal{L}_{j'} \cap Q(S, \bar{r}/2)$ . Therefore, it must be  $z \in \mathcal{L}_{j'} \cap Q_\alpha$  for some  $\alpha \in \mathcal{A}_{j'}$ , i.e.  $z \in \Gamma_{c'}$  for some  $c' = (j', 0, \alpha) > c$ .

**b.** Now consider the case of  $z \in \partial\Gamma_c \cap (S \setminus \Sigma^{3\bar{\gamma}})$  with  $c = (j, 0, \alpha)$  for some  $j = 0, \dots, N$  and  $\alpha \in \mathcal{A}_j$ . In this case, by definition  $\Gamma_c = Q_\alpha \setminus \left( \bigcup_{i < j} \mathcal{L}_i \right)$ . If  $z \in \partial\mathcal{L}_{j-1} \cap Q_\alpha$ , then

$$f(z, u_c(z)) \cdot \mathbf{n}(z) = f(z, q_\alpha) \cdot \mathbf{n}(z) < 0, \quad (51)$$

due to (47) and to the unit normal  $\mathbf{n}(\xi)$  to  $\mathcal{L}_{j-1}$  being parallel to  $\nabla W(\xi)$ , at every point  $\xi$  where  $W$  is differentiable, since  $\partial\mathcal{L}_{j-1}$  is a level set of  $W$ . In other words, the inward-pointing condition is immediately satisfied in  $z$ . If  $z \in \partial\mathcal{L}_{j-1} \cap \partial Q_\alpha$ , one deduces from the fact that the boundary  $\partial\mathcal{L}_{j-1}$  is contained in the union of a finite number of spheres, pairwise non-tangent (see Remark 3.3), that the Clarke tangent cone in  $z$  has nonempty interior. Hence, from (51) there follows that

$$f(z, q_\alpha) \in \overset{\circ}{T}_{\mathcal{L}_{j-1}}^C(z).$$

Combining this fact with  $f(z, q_\alpha)$  pointing inside  $Q_\alpha$ , one concludes that

$$f(z, q_\alpha) \in \overset{\circ}{T}_{\mathcal{L}_{j-1}}^C(z) \cap \overset{\circ}{T}_{Q_\alpha}^C(z) \subseteq \overset{\circ}{T}_{\mathcal{L}_{j-1} \cap Q_\alpha}(z),$$

where we have denoted, like in Section 2, with  $T_\Omega^C$  the Clarke tangent cone to the set  $\Omega$  and with  $T_\Omega$  the Bouligand tangent cone (or contingent cone) to  $\Omega$  and the final inclusion is a consequence of (21). Owing to Remark 2.1, this means that the required inward-pointing condition is satisfied in  $z$ .

Take then

$$\begin{aligned} z &\in \partial Q_\alpha \cap \mathcal{L}_{j'} \cap (S \setminus \Sigma^{3\bar{\gamma}}) \\ &= \partial Q_\alpha \cap \mathcal{L}_{j'} \cap \left( (S_{\bar{r}/2} \cup Q(S, \bar{r}/2)) \setminus \Sigma^{3\bar{\gamma}} \right) \quad j' \geq j, \end{aligned}$$

If  $z \in \partial Q_\alpha \cap \mathcal{L}_{j'} \cap (S_{\bar{r}/2} \setminus \Sigma^{3\bar{\gamma}})$  for some  $j' \geq j$ , then by (49) it must be  $z \in \Gamma_{j',1,\beta}$  for some  $\beta \in \mathcal{B}_{j'}$  and of course  $(j, 0, \alpha) \prec (j', 1, \beta)$ , i.e.  $z \in \Gamma_{c'}$  for some  $c' > c$ . On the other hand, if

$$z \in (\partial Q_\alpha \cap Q(S, \bar{r}/2)) \setminus \bigcup_{\alpha' > \alpha} Q_{\alpha'} \subseteq \partial Q_\alpha \setminus \bigcup_{\alpha' > \alpha} Q_{\alpha'},$$

then the inward-pointing condition is verified because  $u_c = u_\alpha = U^b$ , which was patchy on  $Q(S, \bar{r}/2)$ . Recalling (48), the remaining case is

$$z \in \partial Q_\alpha \cap \mathcal{L}_{j'} \cap \left( \bigcup_{\alpha' > \alpha} Q_{\alpha'} \right) \quad j' \geq j,$$

which implies  $z \in \Gamma_{j',0,\alpha'}$  with  $(j, 0, \alpha) \prec (j', 0, \alpha')$ , and therefore  $z \in \Gamma_{c'}$  for some  $c' > c$ .

**c.** It remains to consider the case  $z \in (\partial \Gamma_c \cap \overset{\circ}{\mathcal{D}}) \setminus S$ . Recalling (45) and the definition of  $\mathcal{D}$ , we have  $z \in (\partial \Gamma_c \cap \overset{\circ}{\mathcal{D}}^b) \setminus S$  and therefore it must be  $c = (j, 0, \alpha)$  for some  $j = 0, \dots, N$  and  $\alpha \in \mathcal{A}_j$ . We also observe that by denoting with  $L_f$  and  $L_W$  Lipschitz constants for  $f$  and  $W$ , respectively, on  $Q(S, \bar{r}/2) + \varepsilon \bar{B}_d$ , by setting

$$M_{Q,f} \doteq \max_{(Q(S, \bar{r}/2) + \varepsilon \bar{B}_d) \times \mathbf{U}} |f(x, u)|,$$

$$0 < \lambda < \min \left\{ \frac{C}{32 L_W L_f}, \frac{C}{32 L_W M_{Q,f}} \right\}$$

and by requiring  $\text{diam } Q_\alpha < \lambda$  in Step 3, from (47) there follows

$$f(z, u_c(z)) \bullet W(z) = f(z, q_\alpha) \bullet W(z) < -\frac{C}{16} < 0.$$

Hence, by repeating the same argument used for the case  $z \in \partial \Gamma_c \cap (S \setminus \Sigma^{3\bar{\gamma}})$  with  $c = (j, 0, \alpha)$ , we obtain the required property also in this case.

Since we have proved that in every case, either the inward pointing condition is satisfied or  $z \in \Gamma_{c'}$  for some  $c' > c$ , this completes the proof that  $U$  is patchy on  $\mathcal{D}$ .

**Step 6.** We prove now that any trajectory  $x(\cdot)$  of (1), corresponding to the control  $U$  and such that  $x(0) \in S \setminus \Sigma^\delta$ , remains inside  $S$  for all times  $t \geq 0$  in its maximal domain of existence  $[0, T_{\max}]$ .

First, observe that it is enough to prove the property for trajectories  $x(\cdot)$  with  $x(0)$  in the interior of  $S$ . Indeed, if  $x(0) \in \partial S$ , then there exists a small  $\tau > 0$  such that in  $]0, \tau]$  a solution exists, because the vector field is patchy, and belongs to the interior of  $S$ , because of (36). Hence, by applying the result on the interior of  $S$  to the solution of (1) with initial datum  $x(\tau)$ , one concludes that the property holds also for trajectories starting from the boundary  $\partial S$ .

Now, take  $x(0) \in \overset{\circ}{S}$  and assume by contradiction there exists  $\bar{t} \in ]0, T_{\max}[$  such that  $x(\bar{t}) \in \partial S$  but  $x(s)$  belongs to the interior of  $S$  for  $0 \leq s < \bar{t}$ . Then,

$$x(\bar{t}) \in \Gamma_{\bar{c}}, \quad \text{where} \quad \bar{c} \doteq c^*(x(\bar{t})) = \max\{c \in \mathcal{C} ; x(\bar{t}) \in \Gamma_c\},$$

and  $u_{\bar{c}} = u_{(j,0,\alpha)} = q_\alpha$  for some index  $\alpha \in \mathcal{A}_j$ , because (45) implies that  $\bigcup_{\beta \in \mathcal{B}} \Gamma_{j,1,\beta} \subseteq S_{\bar{r}/4}$ . Thus,  $f(\cdot, u_{\bar{c}}(\cdot))$  satisfies (36)–(37). Moreover, being  $\Gamma_{\bar{c}}$  an open set and  $U$  a patchy control, there exists  $\tau \in [0, \bar{t}[$  such that for  $s \in ]\tau, \bar{t}]$  one has

$$x(s) \in \Gamma_{\bar{c}}, \quad \dot{x}(s) = f(x(s), u_{\bar{c}}(x(s))) = f(x(s), q_\alpha).$$

Now assume that  $\Delta_S$  is differentiable in  $x(s)$  for a.e.  $s \in ]\tau, \bar{t}]$ . Then, by combining the local decrease property (38), applied in a neighborhood of  $x(\bar{t}) \in \partial S$ , with the boundedness of  $\nabla \Delta_S$  and the continuity of  $f$  in  $\Gamma_{\bar{c}}$ , there exists  $\tau' \in [\tau, \bar{t}]$  such that for a.e.  $s \in [\tau', \bar{t}]$  there holds

$$\begin{aligned} \nabla \Delta_S(x(s)) \cdot f(x(s), q_\alpha) &\leq \nabla \Delta_S(x(s)) \cdot f(x(\bar{t}), q_\alpha) \\ &\quad + \left| \nabla \Delta_S(x(s)) \right| \left| f(x(\bar{t}), q_\alpha) - f(x(s), q_\alpha) \right| \leq -\varepsilon/2 < 0. \end{aligned}$$

In turn, this implies

$$\begin{aligned} d(x(\bar{t}), \overline{\mathbb{R}^d \setminus S}) - d(x(\tau'), \overline{\mathbb{R}^d \setminus S}) &= -\Delta_S(x(\bar{t})) + \Delta_S(x(\tau')) \\ &= -\int_{\tau'}^{\bar{t}} \nabla \Delta_S(x(\sigma)) \cdot \dot{x}(\sigma) d\sigma \geq \frac{\varepsilon}{2} (\bar{t} - \tau') > 0, \end{aligned} \quad (52)$$

which yields

$$d(x(\tau'), \overline{\mathbb{R}^d \setminus S}) < d(x(\bar{t}), \overline{\mathbb{R}^d \setminus S}) = 0.$$

This gives a contradiction because  $x(\tau')$  belongs to  $\overset{\circ}{S}$  and, therefore, its distance from  $\overline{\mathbb{R}^d \setminus S}$  should be positive.

Observe that we may reach the same conclusion even if  $\Delta_S$  is not differentiable along the trajectory  $x(\cdot)$  on a set  $\mathcal{I} \subseteq ]\tau, \bar{t}]$  of positive measure. Indeed, denote with  $\tau' \in ]\tau, \bar{t}]$  a value such that  $x(s) \in \mathcal{N}_{x(\bar{t})}$  for  $s \in [\tau', \bar{t}]$  and  $\mathcal{N}_{x(\bar{t})}$  the neighborhood of  $x(\bar{t})$  in which (38) holds. Then, one can always find an arbitrarily close curve  $x_\alpha(\cdot) \doteq x(\cdot) + \alpha$ ,  $\alpha \in \mathbb{R}^d$  with  $|\alpha| \ll 1$ , where  $\Delta_S$  is differentiable a.e., since otherwise  $\Delta_S$  would turn to be not differentiable on a subset of positive measure of a neighborhood of  $x(\cdot)|_{[\tau', \bar{t}]}$

$$\{x(s) + \rho B_d ; s \in [\tau', \bar{t}], \rho \ll 1\}.$$

Hence, we can repeat the above computation for the variation of  $\Delta_S$  along such a curve  $x_\alpha(\cdot)$ : by  $\dot{x}_\alpha = \dot{x}$  and by the arbitrary closeness of  $x_\alpha$  to  $x(\cdot)$ , we thus reach a contradiction also in this case.

**Step 7.** To complete the proof, it remains to show that  $U$  stabilizes trajectories of (1) to  $\Sigma^\delta$ . By (43) and (47), we know that for all  $x \in \mathcal{D} \cap S$  where  $\nabla W(x)$  is defined, there holds

$$f(x, U(x)) \bullet W(x) < -\frac{C}{8}. \quad (53)$$

Hence, fix any initial datum  $x_o \in S \setminus \Sigma^\delta$  and denote by  $x(\cdot)$  any corresponding Carathéodory solution to (3), defined on its maximal domain  $[0, T_{max}[$ . Notice that, by invariance of the set  $S$  and by the inclusion  $\overline{\Lambda} = \overline{S \setminus \Sigma^{3\bar{\gamma}}} \subset \mathcal{D}$ , it can only happen that either  $x(T_{max}^-) = \lim_{t \rightarrow T_{max}^-} x(t) \in \partial \mathcal{D} \cap \Sigma^{3\bar{\gamma}}$  or that  $T_{max} = +\infty$  and the trajectory remains inside  $\Lambda$  for all times. We claim that the latter case cannot happen.

Indeed, assume  $T_{max} = +\infty$  and that  $x(t) \in \Lambda$  for all  $t \geq 0$ . Introducing  $m_{\bar{\gamma}} \doteq \min_\Lambda \bar{\varphi}$ , we have that  $m_{\bar{\gamma}} > 0$ , because of the Definition 2.3 of strong CLF family, and that  $\min_\Lambda W \geq m_{\bar{\gamma}} > 0$  because of (42). Since we have by (53)

$$W(x(t)) - W(x_o) = \int_0^t f(x(s), U(x(s))) \bullet W(x(s)) ds < -\frac{Ct}{8}, \quad (54)$$

for  $t$  large enough we obtain

$$W(x(t)) < W(x_o) - \frac{Ct}{8} < m_{\bar{\gamma}},$$

which is a contradiction. We can thus conclude that

$$x(T_{max}^-) \in \partial\mathcal{D} \cap \Sigma^{3\tilde{\gamma}} \subset \Sigma^{4\tilde{\gamma}} \subset \Sigma^\delta,$$

and therefore, by continuity, it is well defined

$$T \doteq \inf \left\{ s \in [0, T_{max}[ ; x(\sigma) \in \mathcal{D} \cap \Sigma^\delta \ \forall \sigma \in [s, T_{max}[ \right\},$$

so that for  $t \in [T, T_{max}[$  one has  $x(t) \in \Sigma^\delta$ .

This completes the construction of a feedback control which stabilizes the system to  $\Sigma^\delta$ .  $\diamond$

## 5 Proof of Lemma 3.2

Due to their extensive use in the following proof, we recall that the notations  $S_r$  and  $Q(S, r)$ , for  $r > 0$ , were introduced in (23)–(24) and that for any  $x \in S$  we use  $r(x)$  to denote the quantity  $d(x, \overline{\mathbb{R}^d \setminus S})$ .

**Step 1.** First of all, let  $r_o > 0$  be as in Lemma 2.1,  $v: Q(S, r_o) \rightarrow \mathbb{R}^d$  be as in Lemma 2.2 and  $\tilde{\varepsilon} > 0$  be as in Lemma 2.3. Fix  $\varepsilon \in ]0, \tilde{\varepsilon}/2[$  and  $\tilde{r} \in ]0, r_o[$ . By uniform continuity of the function  $v$  on  $Q(S, r_o)$  and of the function  $f$  on  $Q(S, r_o) \times \mathbf{U}$ , there exists  $\delta > 0$  such that for all  $x, y \in Q(S, r_o)$  and  $u \in \mathbf{U}$  one has

$$\begin{aligned} |x - y| < \delta &\implies |v(x) - v(y)| < \frac{\varepsilon}{2}, \\ &|f(x, u) - f(y, u)| < \frac{\varepsilon}{2}. \end{aligned} \tag{55}$$

Notice that, in view of (34), it is not restrictive to assume  $\delta < r_o - \tilde{r}$  and that, in view of the required bound on the diameter of the domains, we can assume also  $\delta < \lambda/2$  for any fixed constant  $\lambda > 0$ . Finally, we remark that, for any radius  $R > 0$ , one always has

$$|v - v'| < \frac{R}{2} \implies \mathcal{W}(v', R/2) \subseteq \mathcal{W}(v, R). \tag{56}$$

**Step 2.** In order to define the patchy feedback, we start with the construction of suitable neighborhoods  $\Gamma^x$  around each point  $x \in Q(S, \tilde{r})$ . Recalling that by Lemma 2.2 we have  $v$  Lipschitz continuous and  $v \neq 0$  on  $Q(S, r_o)$ , we set

$$M_v \doteq \max_{Q(S, r_o)} |v(x)| < +\infty, \quad m_v \doteq \min_{Q(S, r_o)} |v(x)| > 0,$$

and fix  $\beta$  such that

$$0 < \beta < \min \left\{ 1, \frac{8\delta}{3(2M_v\tilde{\varepsilon} + \tilde{\varepsilon}^2)} \right\}.$$

Consider now the wedge  $\mathcal{W}(v(x), \tilde{\varepsilon}/2)$  and recall that for points  $z \in \partial^- \mathcal{W}(v(x), \tilde{\varepsilon}/2)$  there holds (22). Then, a similar estimate holds for points

$$z \in \partial^- \left( \beta \mathcal{W}(v(x), \tilde{\varepsilon}/2) \right) = \beta \partial^- \mathcal{W}(v(x), \tilde{\varepsilon}/2).$$

Namely, every point in  $z \in \partial^- \left( \beta \mathcal{W}(v(x), \tilde{\varepsilon}/2) \right)$  has an uniformly positive distance from  $\mathbb{R}^d \setminus \mathcal{W}(v(x), \tilde{\varepsilon})$ , that we denote with  $\rho_x$ . Since the axis of  $\mathcal{W}(v(x), \tilde{\varepsilon}/2)$  always satisfies  $|v(x)| \geq m_v > 0$  for  $x \in Q(S, r_o)$ , there also is  $\rho_o > 0$  such that

$$\rho_o \leq \rho_x \quad \forall x \in Q(S, r_o),$$

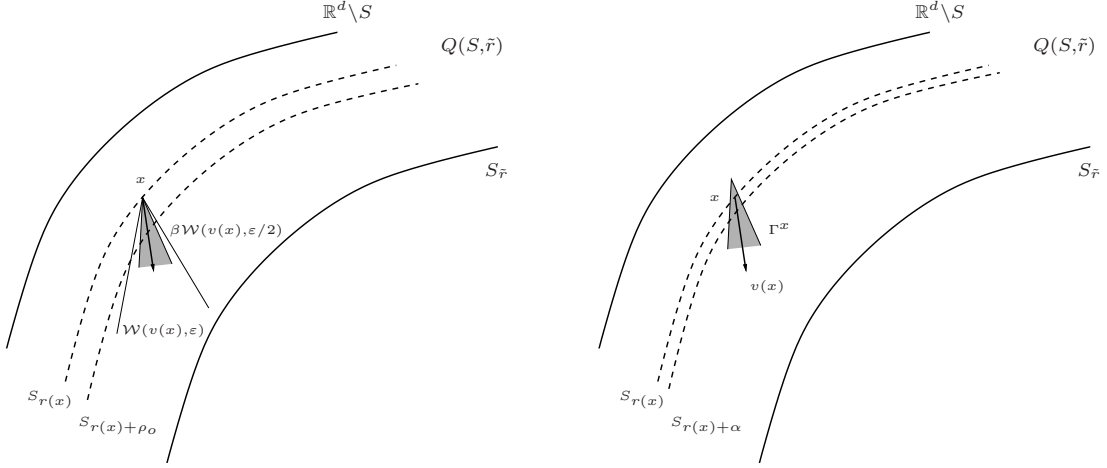


Figure 4: Left: The rescaled wedge  $x + \beta\mathcal{W}(v(x), \varepsilon/2)$  is well inside the wedge  $x + \mathcal{W}(v(x), \varepsilon)$ . Right: The domain  $\Gamma^x$  is obtained by a small translation of  $x + \beta\mathcal{W}(v(x), \varepsilon/2)$ .

and hence (see Figure 4 left)

$$0 < \rho_o \leq d(z, \overline{\mathbb{R}^d \setminus \mathcal{W}(v(x), \tilde{\varepsilon})}) \quad \forall z \in \partial^- \left( \beta\mathcal{W}(v(x), \tilde{\varepsilon}/2) \right) \quad \forall x \in Q(S, r_o).$$

We finally fix  $\alpha$  such that

$$0 < \alpha < \min \left\{ \frac{\delta}{3}, \frac{\rho_o}{2}, \frac{\beta}{2} \right\}.$$

and define, for every  $x \in Q(S, \tilde{r})$

$$\Gamma^x \doteq x - \alpha \frac{v(x)}{|v(x)|} + \beta \overset{\circ}{\mathcal{W}}(v(x), \tilde{\varepsilon}/2).$$

In other words, we first rescale the (open) wedge  $x + \overset{\circ}{\mathcal{W}}(v(x), \tilde{\varepsilon}/2)$  by a suitable factor  $\beta$  and then we slightly shift it along the direction  $-v(x)$ , in order to obtain an open neighborhood of  $x$ . The particular choice of  $\alpha, \beta$  ensures that for every  $x \in Q(S, \tilde{r})$  there holds

$$\Gamma^x \subseteq B(x, \delta). \quad (57)$$

Indeed, given  $\xi \in \Gamma^x$  there exist  $s, \sigma \in [0, \tilde{\varepsilon}/2[$  and  $z \in B_d$  such that

$$\xi = x - \alpha \frac{v(x)}{|v(x)|} + \beta s(v(x) + \sigma z),$$

and, hence,

$$|x - \xi| \leq \alpha + \beta s(|v(x)| + \sigma) \leq \alpha + \frac{\beta}{4}(2M_v \tilde{\varepsilon} + \tilde{\varepsilon}^2) < \delta.$$

Moreover, if we split the boundary  $\partial\Gamma^x$  into its “lower” and “upper” parts, by setting

$$\partial^-\Gamma^x \doteq x - \alpha \frac{v(x)}{|v(x)|} + \partial^-\left(\beta\mathcal{W}(v(x), \tilde{\varepsilon}/2)\right), \quad \partial^+\Gamma^x \doteq \partial\Gamma^x \setminus \partial^-\Gamma^x,$$

the choice of the parameters  $\alpha$  and  $\beta$  also ensures that the lower boundary  $\partial^-\Gamma^x$  is well inside the wedge  $x + \mathcal{W}(v(x), \tilde{\varepsilon})$  for every  $x \in Q(S, \tilde{r})$ . Indeed, for  $z \in \partial^-\Gamma^x$  we have

$$\begin{aligned} d\left(z, \overline{\mathbb{R}^d \setminus (x + \mathcal{W}(v(x), \tilde{\varepsilon}))}\right) &\geq d\left(z + \alpha \frac{v(x)}{|v(x)|}, \overline{\mathbb{R}^d \setminus (x + \mathcal{W}(v(x), \tilde{\varepsilon}))}\right) - \alpha \\ &> \rho_o - \frac{\rho_o}{2} > 0. \end{aligned}$$

In particular, fixing a point  $x \in Q(S, \tilde{r})$ , the wedge  $x + \mathcal{W}(v(x), \tilde{\varepsilon})$  is contained in  $S_{r(x)}$  by Lemma 2.3, and hence for every  $z \in \partial^-\Gamma^x$  there holds

$$d\left(z, \overline{\mathbb{R}^d \setminus S_{r(x)}}\right) \geq d\left(z, \overline{\mathbb{R}^d \setminus (x + \mathcal{W}(v(x), \tilde{\varepsilon}))}\right) > \frac{\rho_o}{2} > \alpha,$$

which ensures that

$$r(z) \geq r(x) + \alpha \quad \forall z \in \partial^-\Gamma^x. \quad (58)$$

**Step 3.** Recalling (25), for any choice of a point  $x \in Q(S, \tilde{r})$  there exists  $u_x \in \mathbf{U}$  such that  $v(x) = f(x, u_x)$ . Now, we claim that by choosing  $U(\cdot) \equiv u_x$  in  $\Gamma^x$  properties (35), (36) and (37) are satisfied on each domain  $\Gamma^x$ . For all  $\eta \in \Gamma^x \cap Q(S, \tilde{r})$ , inclusion (57) implies  $|\eta - x| < \delta$ , and hence by (55) there holds

$$|f(\eta, u_x) - v(\eta)| \leq |f(\eta, u_x) - f(x, u_x)| + |v(x) - v(\eta)| < \varepsilon < \tilde{\varepsilon}/2.$$

This already gives (35). Moreover, recalling (56), we deduce that for all  $\eta \in \Gamma^x \cap Q(S, \tilde{r})$  there holds

$$\begin{aligned} \mathcal{W}(f(\eta, u_x), \varepsilon) &\subseteq \mathcal{W}(f(\eta, u_x), \tilde{\varepsilon}/2) \subseteq \mathcal{W}(v(\eta), \tilde{\varepsilon}), \\ \mathcal{W}(-f(\eta, u_x), \varepsilon) &\subseteq \mathcal{W}(-f(\eta, u_x), \tilde{\varepsilon}/2) \subseteq \mathcal{W}(-v(\eta), \tilde{\varepsilon}), \end{aligned}$$

In turn, with Lemma 2.3, this implies for all  $y \in \{\eta + \tilde{\varepsilon}B_d\} \cap S_{r(\eta)}$

$$y + \mathcal{W}(f(\eta, u_x), \varepsilon) \subseteq y + \mathcal{W}(v(\eta), \tilde{\varepsilon}) \subseteq S_{r(\eta)},$$

and for all  $y \in \{\eta + \tilde{\varepsilon}B_d\} \setminus \overset{\circ}{S}_{r(x)}$

$$y + \mathcal{W}(-f(\eta, u_x), \varepsilon) \subseteq y + \mathcal{W}(-v(\eta), \tilde{\varepsilon}) \subseteq \mathbb{R}^d \setminus S_{r(\eta)},$$

i.e. (36) and (37).

Finally, we claim that the vector  $f(\eta, u_x)$  is always inward-pointing at points  $\eta \in \partial^+\Gamma^x \cap Q(S, \tilde{r})$ . Indeed, once again (55) and (57) give  $|f(\eta, u_x) - v(x)| < \varepsilon/2 < \tilde{\varepsilon}/4$  and with (56) this implies  $\eta + cf(\eta, u_x) \in \eta + \mathcal{W}(v(x), \tilde{\varepsilon}/2)$  for some  $0 < c \leq 1$ , proving the inward pointing condition at  $\eta$ .

**Step 4.** We now replace our sets  $\Gamma^x$  with the slightly smaller ones defined by

$$\Omega^x \doteq \Gamma^x \setminus S_{r(x)+\alpha},$$

where  $\alpha$  is the parameter chosen in the construction of the domains  $\Gamma^x$ . Of course,  $\Omega^x$  is still an open neighborhood of the point  $x$ , like  $\Gamma^x$  was. Moreover, recalling (58), we observe that by removing  $S_{r(x)+\alpha}$  we have cut away a part of  $\bar{\Gamma}^x$  which contained the whole “lower” boundary  $\partial^-\Gamma^x$ . Hence, we can divide the boundary of  $\Omega$  in

$$\partial^+\Omega^x \doteq \partial^+\Gamma^x \setminus S_{r(x)+\alpha}, \quad \partial^-\Omega^x \doteq \partial\Omega^x \setminus \partial^+\Omega^x = \Gamma^x \cap \partial S_{r(x)+\alpha},$$

and now we have  $r(z) = r(x) + \alpha$  for all  $z \in \partial^- \Omega^x$ . Moreover, at points  $\eta \in \partial^+ \Omega^x \cap Q(S, \tilde{r}) \subseteq \partial^+ \Gamma^x \cap Q(S, \tilde{r})$  the vector  $f(\eta, u_x)$  is still inward-pointing.

Now, by compactness of  $Q(S, \tilde{r})$ , there exists a finite set of points  $x_1, \dots, x_N \in Q(S, \tilde{r})$  such that

$$Q(S, \tilde{r}) \subseteq \bigcup_{i=1}^N \Omega^{x_i}.$$

We re-label the sets as follows

$$\Omega_{r(x_i), i} \doteq \Omega^{x_i},$$

and we order the collection  $\{\Omega_{r(x_i), i}\}$  with lexicographic order (19), i.e.

$$(r(x_i), i) \prec (r(x_j), j) \quad \text{iff} \quad \text{either } r(x_i) < r(x_j) \text{ or } r(x_i) = r(x_j) \text{ and } i < j.$$

The advantage of this choice is that we can easily prove that points of  $\partial^- \Omega^{x_i}$ , i.e. of the “lower” boundary of  $\Omega^{x_i}$ , belong either to  $S_{\tilde{r}}$  or to another domain  $\Omega_{r(x_j), j}$  with larger index. Namely, fix  $i \in \{1, \dots, N\}$  and let  $z \in \partial^- \Omega^{x_i}$ . Then, if  $r(z) < \tilde{r}$  there must be an index  $j \in \{1, \dots, N\}$  such that  $z \in \Omega^{x_j}$ . Assuming now that

$$(r(x_i), i) \not\prec (r(x_j), j)$$

we would have, in particular, that  $r(x_i) \geq r(x_j)$  and therefore

$$d(z, \mathbb{R}^d \setminus S) = r(x_i) + \alpha \geq r(x_j) + \alpha \implies z \in S_{r(x_j) + \alpha},$$

which means  $z \notin \Omega^{x_j} = \Gamma^{x_j} \setminus S_{r(x_j) + \alpha}$ , i.e. a contradiction.

**Step 5.** We claim that by setting  $\mathcal{D} \doteq \bigcup_{j=1}^N \Omega_{r(x_j), j}$  and  $U: \mathcal{D} \rightarrow \mathbf{U}$  the piecewise constant feedback law defined by

$$U(\xi) \doteq u_{x_i} \quad \text{for all } \xi \in \Omega_{r(x_i), i} \setminus \bigcup_{(r(x_i), i) \prec (r(x_j), j)} \Omega_{r(x_j), j}$$

we obtain a patchy feedback control  $U$  which has the required properties. Indeed, the choice of  $\delta$  in Step 1 ensures that

$$\text{diam } \Omega_{r(x_j), j} \leq \text{diam } B(x_j, \delta) \leq \lambda, \quad \mathcal{D} \subseteq Q(S, \tilde{r}) + (r_o - \tilde{r})\overline{B_d}.$$

Moreover, in Step 3 we have proved that (35), (36) and (37) hold for all  $y \in Q(S, \tilde{r})$ . Finally, Step 4 proves both that  $f(\cdot, U(\cdot))$  is inward pointing at points of  $\partial^+ \Omega_{r(x_j), j}$  and that points of  $\partial^- \Omega_{r(x_j), j}$  do not need to be taken into account because they belong either to a patch with larger index  $(r', j')$  or to  $\partial \mathcal{D} \cap S_{\tilde{r}}$  (see Remark 2.3). Hence,  $U$  is a patchy feedback control and the proof is complete.  $\diamond$

## 6 Extensions and remarks

### 6.1 $S$ -restricted dynamics

Our main result has been stated and proved for a control system (1) whose vector field  $f$  is defined in the whole  $\mathbb{R}^d \times \mathbf{U}$ . In many applications to economy and engineering, though, the dynamics could have no meaning or even break down completely when  $x \notin S$ . As such we want to stress that Theorem 1 can be also applied to the case of a control system (1) whose dynamics is given by a function  $f: S \times \mathbf{U} \rightarrow \mathbb{R}^d$  not defined for  $x \notin S$ .

Indeed, in the same spirit of [11, 12], we can extend  $f$  to a globally Lipschitz continuous function  $\tilde{f}: \mathbb{R}^d \times \mathbf{U} \rightarrow \mathbb{R}^d$  by defining  $\tilde{f} = (\tilde{f}_1, \dots, \tilde{f}_d)$  as follows

$$\tilde{f}_i(x, u) \doteq \min_{y \in S} \{f_i(y, u) + L_f |x - y|\}. \quad (59)$$

Now, assuming that the vector field  $f$  satisfies **(F1)**, **(F2)** and **(F3)** on its domain  $S$ , then also  $\tilde{f}$  satisfies **(F1)** and **(F2)** on the whole  $\mathbb{R}^d$ . However,  $\tilde{f}$  might fail to satisfy **(F3)** outside  $S$ . Luckily, this is not a problem for our result: we needed convexity of  $f(x, \mathbf{U})$  to ensure that the function  $v(x)$  in Lemma 2.2 equals  $f(x, u_x)$  for a suitable admissible control value  $u_x \in \mathbf{U}$ . As such, **(F3)** is only necessary inside  $S$ , and the rest of our construction can be applied to the dynamics

$$\dot{x} = \tilde{f}(x, u), \quad (60)$$

with no significant change.

## 6.2 Unbounded constraints

Another assumption that can be relaxed in our main result is the compactness of the constraint set  $S$ . Indeed, as it has been done for sample-and-hold trajectories in [12], one can require  $S$  to only be a closed set. In the latter case, it is possible to prove the following

**Theorem 2** *Let  $S$  be a set satisfying **(S1)** and **(S2)**, except for compactness in **(S1)** replaced by closedness, and let  $\Sigma$  be any closed set such that  $S \cap \Sigma \neq \emptyset$ . Assume that for all bounded sets of initial data  $\mathcal{B}$ , the trajectories of the system (1) starting from  $\mathcal{B}$  are open loop controllable to  $\Sigma$  remaining inside  $S$ . Then, for every fixed bounded set  $\mathcal{B}$  there exists a patchy feedback control  $U = U_{\mathcal{B}}(x)$  which makes (1) practically stable to  $\Sigma$  subject to the constraint  $S$ .*

To prove the theorem above, notice that the results about wedged sets and strong CLF families given in Section 2 still hold under the relaxed assumptions of Theorem 2 (see [12, 13]). Therefore, for the fixed bounded set  $\mathcal{B}$ , one can simply take a large enough ball  $K = kB_d$  such that  $\mathcal{B} \subseteq K$  and  $(\overline{S \cap K}) \cap \Sigma \neq \emptyset$ , and apply Theorem 1 to the smaller constraint set  $S' \doteq \overline{S \cap K}$ . In this way, for every  $\delta > 0$  we obtain a patchy feedback control such that trajectories starting from  $\mathcal{B}$  remain inside  $S$  for all positive times and eventually reach  $(S \cap K) \cap \Sigma^\delta \subseteq \Sigma^\delta$ , as required.

## 6.3 Robustness

One of the main advantages of using patchy controls  $U(x)$  and Carathéodory solutions for (1) over allowing arbitrarily discontinuous controls and weaker concepts of solutions like sample-and-hold trajectories (see [11, 12, 13]), is that stronger robustness properties can be proved with almost no efforts.

It is indeed well known that whenever the vector field

$$g(x) = f(x, U(x)),$$

is a patchy vector field in the sense of Definition 2.1, then the set of Carathéodory solution is robust w.r.t. both internal and external perturbations (see [1, 2, 4]) without any additional assumption on the feedback control  $U$ . This represents a noticeable improvement compared to the construction obtained in [11, 12, 13] through sample-and-hold and Euler solution, which only achieves the same robustness by requiring a “reasonable uniformity” in the time discretization.

This additional robustness holds also in the case of a constrained dynamics. Namely, the following theorem holds.



**Theorem 3** Assume that the system (1) satisfies open loop  $S$ -constrained controllability to  $\Sigma$ , where  $S$  is a set satisfying **(S1)** and **(S2)** and  $\Sigma$  is any closed set such that  $S \cap \Sigma \neq \emptyset$ . Then, for every  $\delta > 0$  there exist  $T > 0$ ,  $\chi > 0$  and a patchy feedback control  $U = U(x)$ , defined on an open domain  $\mathcal{D}$  with  $S \setminus \Sigma^\delta \subseteq \mathcal{D}$ , so that the following holds. Given any pair of maps  $\zeta \in \mathbf{BV}([0, T], \mathbb{R}^d)$  and  $d \in \mathbf{L}^1([0, T], \mathbb{R}^d)$  such that

$$\|\zeta\|_{\mathbf{BV}} \doteq \|\zeta\|_{\mathbf{L}^1} + \text{Tot.Var.}(\zeta) < \chi, \quad \|d\|_{\mathbf{L}^1} < \chi,$$

and any initial datum  $x_o \in S \setminus \Sigma^\delta$ , for every Carathéodory solution  $t \mapsto y(t)$ , defined for  $t \in [0, T]$ , of the perturbed Cauchy problem

$$\dot{y} = f(y, U(y + \zeta)) + d, \quad y(0) = x_o, \quad (61)$$

one has

$$y(t) \in S \quad \forall t \in [0, T], \quad \text{and} \quad y(T) \in \Sigma^\delta.$$

We do not include the explicit proof of this theorem, since it follows very closely the one of Theorem 3.4 in [2], by combining Theorem 1 with the general robustness results for patchy vector fields in presence of impulsive perturbations, proved by Ancona and Bressan in [2].

## 6.4 A new construction for unconstrained stabilizing patchy feedbacks

Among possible applications of Lemma 3.1, we mention here the construction of a practically stabilizing feedback for control systems (1) which are GAC to the origin and with no constraints on the dynamics. This result was first proved in [1], by means of a different construction, relying on open-loop controls rather than on Lyapunov functions.

We recall that an unconstrained control system (1) is said to be *globally asymptotically controllable* (GAC) to 0 if for every initial datum there exists an open-loop control, with values in  $\mathbf{U}$ , which steers the corresponding solutions to the origin, and if solutions starting sufficiently close to 0 remain close to 0 for all times (see e.g. [1, 14, 17]). By the results in [14], we know that if (1) is GAC to the origin, then there exists a control Lyapunov function  $V: \mathbb{R}^d \rightarrow [0, +\infty[$  which is locally semiconcave on  $\mathbb{R}^d \setminus \{0\}$  and such that

$$\min_{\omega \in \mathbf{U}} \{ \nabla V(x) \cdot f(x, \omega) \} \leq -V(x)$$

for all  $x \in \mathbb{R}^d \setminus \{0\}$  where  $\nabla V$  exists. Now, fixed any compact set of initial data  $K \subseteq \mathbb{R}^d$  and a target ball  $\varepsilon' B_d$  around the origin, we prove that there exists a patchy feedback which stabilizes to  $\varepsilon' B_d$  every trajectory starting from  $x_o \in K$ .

First of all, we apply Lemma 3.1 to  $\Omega = \mathbb{R}^d \setminus \{0\}$ ,

$$\Lambda = \left\{ \xi \in \mathbb{R}^d ; V(\xi) \leq \max_K V + 1 \right\} \setminus \{0\},$$

$h(x) = V(x)$  and  $\rho = \varepsilon = \delta/2$  for a suitable  $\delta \in ]0, 1]$  such that

$$\mathcal{E} \doteq \{ \xi \in \mathbb{R}^d ; V(\xi) \leq \delta \} \subset \varepsilon' B_d.$$

Notice that such a value  $\delta$  exists because  $V$  is proper. Hence, we obtain a piecewise quadratic function  $W: \mathbb{R}^d \rightarrow \mathbb{R}$  and a patchy feedback control  $U: \mathcal{D} \rightarrow \mathbf{U}$ , defined on an open domain

$$\mathcal{D} \supseteq \left\{ \xi \in \mathbb{R}^d ; \delta < V(\xi) \leq \max_K V + 1 \right\} \supset \Lambda \setminus \varepsilon' B_d,$$

such that

$$V(x) \leq W(x) \leq V(x) + \frac{\delta}{2},$$

for all  $x \in \Lambda$ , and

$$f(x, U(x)) \bullet W(x) \leq -V(x) + \frac{\delta}{2},$$

for all  $x \in \mathcal{D} \setminus \mathcal{E}$ .

We claim that  $W$  decreases along the solutions corresponding to  $U$ , until the trajectories stabilizes inside  $\varepsilon' B_d$ . Indeed, fixed an initial datum  $x_o \in K \setminus \varepsilon' B_d$ , we denote with  $x(\cdot)$  any corresponding Carathéodory solution starting from  $x_o$ . In any interval  $[t_o, t_1]$ , there holds

$$\begin{aligned} W(x(t_1)) &= W(x(t_o)) + \int_{t_o}^{t_1} f(x(\sigma), U(x(\sigma))) \bullet W(x(\sigma)) d\sigma \\ &\leq W(x(t_o)) + \int_{t_o}^{t_1} \left( -V(x(\sigma)) + \frac{\delta}{2} \right) d\sigma < W(x(t_o)) - \frac{\delta}{2} (t_1 - t_o) < W(x(t_o)) \end{aligned} \quad (62)$$

where we have used the fact that  $x(\sigma) \notin \mathcal{E}$  and hence  $V(x(\sigma)) > \delta$  for  $\sigma \in [t_o, t_1]$ .

Hence, from (62) and  $W(x_o) \leq V(x_o) + 1$ , we can deduce that the trajectory  $x(\cdot)$  starting at  $x_o \in \Lambda \subseteq \mathcal{D}$  remains in the bounded set  $\Lambda$  for all times in its maximal domain  $[0, T_{max}[$  and, in particular, that  $|x(s)| \not\rightarrow +\infty$  and  $x(s) \not\rightarrow \partial\mathcal{D} \setminus \mathcal{E} \subset \partial\mathcal{D} \setminus \Lambda$ , as  $s \rightarrow T_{max}$ . Therefore, either the trajectory  $x(\cdot)$  remains in  $\Lambda \setminus \mathcal{E}$  for all times and  $T_{max} = +\infty$ , or  $x(T_{max}^-) = \lim_{t \rightarrow T_{max}^-} x(t) \in \partial\mathcal{D} \cap \mathcal{E}$ . In the former case, we would get a contradiction because for  $s$  large enough

$$V(x(s)) \leq W(x(s)) < W(x(t_o)) - \frac{\delta}{2} (s - t_o) < \frac{\delta}{2},$$

which implies  $x(s) \in \mathcal{E}$ . We thus have that

$$x(T_{max}^-) \in \partial\mathcal{D} \cap \mathcal{E} \subset \varepsilon' B_d,$$

and therefore, by continuity, it is well defined

$$T \doteq \inf \{ s \in [0, T_{max}[ ; x(\sigma) \in \mathcal{D} \cap \varepsilon' B_d \ \forall \sigma \in [s, T_{max}[ \} ,$$

so that for  $t \in [T, T_{max}[$  one has  $x(t) \in \varepsilon' B_d$ , i.e.  $U$  stabilizes Carathéodory trajectories to  $\varepsilon' B_d$  as claimed.

We conclude by remarking that, as in [1], once a practically stabilizing patchy (to the origin) has been constructed, one can also construct a stabilizing patchy (to the origin), by suitably gluing together the controls  $U_n$  which realize practical stabilization to  $\frac{1}{n} B_d$ .

## 6.5 Conclusions and open problems

In this paper we have positively solved the problem of practical stabilization of a constrained dynamics through feedback controls. As expected, the control is in general discontinuous but it is possible to select the discontinuity in a suitable way so to obtain a patchy feedback control, which in turn ensures the existence of Carathéodory solutions of the closed loop system for positive times, and the robustness of the feedback with respect to both inner and outer disturbances.

The problem that remains open and that is currently under investigation is the existence of nearly optimal patchy feedbacks for a constrained dynamics. In the unconstrained case, it is known that nearly optimal patchy controls exists (see [3, 7]), but in the constrained case the only available result is the one contained in [11], involving general discontinuous controls and

Euler solutions. In view of the further robustness properties enjoyed by patchy controls, it would be of interest to provide a similar construction in terms of patchy feedbacks and Carathéodory trajectories.

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## Appendix: Proof of Lemma 3.1

The proof is divided in several steps. First, in Step 1–Step 3 we define an approximate function  $W$  with the required properties, then in Step 4–Step 8 we define the patchy feedback control  $U$ , and finally in Step 9 we observe that such a construction allows to find a constant  $\sigma > 0$  and a subset of indices  $\mathcal{B} \subseteq \mathcal{A}$  with the properties stated in Remark 3.2.

**Step 1.** Let  $V$  be a given locally semiconcave function and  $h$  be a given continuous function such that (29) holds. Fix  $\Lambda$  bounded set in the domain of  $V$ ,  $\rho > 0$  and  $\varepsilon$  such that  $0 < \varepsilon < \max_{\Lambda} h(x)$ . Fix also two positive constants  $\lambda, \lambda'$  which are required for properties (iii) and (iv). By semiconcavity of  $V$ , there exists  $\kappa > 0$  such that, for any  $y, y' \in \Lambda$ , one has

$$V(y') \leq V(y) + \mathbf{w} \cdot (y' - y) + \kappa \frac{|y' - y|^2}{2}, \quad (63)$$

for some vector  $\mathbf{w} \in D^+V(y)$  in the superdifferential of  $V$  at the point  $y$ , i.e. for some vector  $\mathbf{w}$  such that

$$\limsup_{\xi \rightarrow y} \frac{V(\xi) - V(y) - \mathbf{w} \cdot (\xi - y)}{|\xi - y|}.$$

Moreover,  $V$  is Lipschitz continuous on its domain  $\Omega$  and we denote with  $L_V$  a Lipschitz constant for  $V$ , namely

$$V(x) - V(y) \leq L_V |x - y| \quad \forall x, y \in \Omega. \quad (64)$$

Hence, by Rademacher's theorem,  $V$  is differentiable almost everywhere in  $\Omega$ .

Finally, notice that given any  $\delta_o > 0$ , we can choose finitely many points  $y_1, \dots, y_q \in \Lambda$  such that  $\nabla V(y_i)$  is well defined for each  $i = 1, \dots, q$ , and moreover

$$\bar{\Lambda} \subseteq \bigcup_{i=1}^q B(y_i, \delta_o). \quad (65)$$

Therefore, we can define an approximate value function, depending on the choice of  $\delta_o$ ,

$$W(x) \doteq \min\{W_1(x), \dots, W_q(x)\},$$

where

$$W_i(x) \doteq V(y_i) + \nabla V(y_i) \cdot (x - y_i) + \kappa |x - y_i|^2. \quad (66)$$

A couple of remarks are in order. First of all, each function  $W_i$  is quadratic, and hence differentiable. Moreover, there holds

$$\nabla W_i(x) = \nabla V(y_i) + 2\kappa(x - y_i), \quad (67)$$

and  $\nabla W_i(y_i) = \nabla V(y_i)$ .

**Step 2.** We claim that, by choosing  $\delta_o = \delta_o(\rho, \varepsilon) > 0$  sufficiently small in the definition of  $W$ , we can obtain the following. For all  $x \in \Lambda$  there hold:

$$V(x) \leq W(x) \leq V(x) + \rho \quad (68)$$

$$\min_{u \in \mathbf{U}} \{\nabla W_i(x) \cdot f(x, u)\} + h(x) \leq \varepsilon/4 \quad \text{whenever } W_i(x) = W(x). \quad (69)$$

Indeed, the first inequality in (68) follows from (63). Next, since  $f$  and  $h$  are continuous and  $\mathbf{U}$  is compact in  $U$ , we can find  $\delta_1 \in ]0, 1]$  such that the following conditions hold. If  $x \in \Lambda$ ,  $\mathbf{w} = \nabla V(y)$  exists and

$$|\mathbf{w}' - \mathbf{w}| \leq 2\kappa\delta_1, \quad |x - y| \leq \delta_1,$$

then (29), i.e.

$$\min_{u \in \mathbf{U}} \{\mathbf{w} \cdot f(y, u)\} + h(y) \leq 0,$$

implies

$$\min_{u \in \mathbf{U}} \{\mathbf{w}' \cdot f(x, u)\} + h(x) \leq \varepsilon/4. \quad (70)$$

We now choose  $\delta_o > 0$  such that

$$2L_V\delta_o + \kappa\delta_o^2 \leq \min \left\{ \rho, \frac{\kappa\delta_1^2}{2} \right\},$$

and select the points  $y_1, \dots, y_q$  to cover  $\bar{\Lambda}$  as in (65).

To prove that such a choice of  $\delta_o$  gives a function  $W$  with the required properties, fix any  $x \in \Lambda$  and let  $j$  be an index such that  $|x - y_j| \leq \delta_o$ . Recalling the Lipschitz condition (64) we find

$$\begin{aligned} W(x) &\leq V(y_j) + |\nabla V(y_j)| |x - y_j| + \kappa |x - y_j|^2 \leq V(x) + 2L_V |x - y_j| + \kappa |x - y_j|^2, \\ W(x) - V(x) &\leq \min \left\{ \rho, \frac{\kappa\delta_1^2}{2} \right\}. \end{aligned} \quad (71)$$

This already yields (68). Comparing (63) with (66), we notice that for all  $x \in \Lambda$  and all  $j \in \{1, \dots, q\}$  there holds

$$W_j(x) - V(x) \geq \kappa \frac{|x - y_j|^2}{2}.$$

Hence, from (71), there follows

$$|x - y_j| \leq \delta_1, \quad \text{whenever } W_j(x) = W(x). \quad (72)$$

Observing that, if  $|x - y_j| \leq \delta_1$ ,

$$|\nabla W_j(x) - \nabla W_j(y_j)| = 2\kappa |x - y_j| \leq 2\kappa\delta_1,$$

from (70) and (72) we deduce the inequality (69). This establishes our claim.

**Step 3.** By observing that for any choice of  $\lambda > 0$  in Step 1, we can always assume  $\delta_1 \leq \min\{\lambda, \frac{\lambda}{2\kappa}\}$  in Step 2, we immediately obtain that if  $W(x) = W_j(x)$ , then the point  $y_j$  used to construct  $W_j$  has the property (33). Indeed, exactly like in Step 2, one has  $|x - y_j| \leq \lambda$  and

$$|\nabla W_j(x) - \nabla V(y_j)| = |\nabla W_j(x) - \nabla W_j(y_j)| \leq 2\kappa\delta_1 \leq \lambda.$$

This ensures that  $W$  satisfies (iii).

**Step 4.** Now we move to the construction of the feedback control  $U$ . We set

$$\mathcal{E} \doteq \{x \in \Lambda ; h(x) \leq \varepsilon\} . \quad (73)$$

First of all, we need an estimate on the size of level sets of the functions  $W_i$  over  $\Lambda \setminus \mathcal{E}$ . Since (66) implies

$$W_i(x) = \kappa|x - x_i|^2 + V(y_i) - \frac{|\nabla V(y_i)|^2}{4\kappa} ,$$

with  $x_i = y_i - \nabla V(y_i)/2\kappa$ , it is clear that all level sets where  $W_i$  is constant are spheres. Indeed, for any given constant  $c > 0$ , we can write

$$\{x \in \mathbb{R}^d ; W_i(x) = c\} = \{x \in \mathbb{R}^d ; |x - x_i| = r_i\} ,$$

for a suitable radius  $r_i > 0$ . We want to prove that  $r_i$  is in fact bounded above and below, for levels which intersect  $\Lambda \setminus \mathcal{E}$ .

For each  $i = 1, \dots, q$ , consider the set

$$\Lambda_i \doteq \{x \in \Lambda \setminus \mathcal{E} ; W_i(x) = W(x)\} .$$

We show that there exists a maximum radius  $r_{max}$  and a minimum radius  $r_{min} > 0$  such that, fixed  $x \in \Lambda_i$ , the level set  $\{\xi \in \mathbb{R}^d ; W_i(\xi) = W_i(x)\}$  is a sphere of center  $x_i$  and radius  $r_i$  with

$$0 < r_{min} \leq r_i \leq r_{max} . \quad (74)$$

Indeed, by (66) and (72), we have for every  $x \in \Lambda$

$$|\nabla W_i(x)| \leq |\nabla W_i(y_i)| + 2\kappa|x - y_i| \leq L_V + 2\kappa\delta_1 \leq L_V + 2\kappa .$$

At the same time, when  $x \in \Lambda \setminus \mathcal{E}$ , from (69) and (73) there follows

$$\max_{u \in \mathbf{U}} |\nabla W_i(x)| |f(x, u)| > h(x) - \frac{\varepsilon}{4} > \frac{3\varepsilon}{4} .$$

Hence, calling

$$M_f \doteq \max_{x \in \Lambda, u \in \mathbf{U}} |f(x, u)| \leq C_f(1 + \text{diam } \Lambda) , \quad (75)$$

we deduce

$$|\nabla W_i(x)| > \frac{4\varepsilon}{4M_f} = \hat{C} .$$

Therefore, recalling the choice of  $x_i$  and (67), fixed  $x \in \Lambda_i \subseteq \Lambda \setminus \mathcal{E}$  and chosen  $\xi$  with  $W_i(\xi) = W_i(x)$ , one has

$$|\xi - x_i| = |x - x_i| = \frac{|\nabla W_i(x)|}{2\kappa} \in \left] \frac{\hat{C}}{2\kappa}, 1 + \frac{L_V}{2\kappa} \right[ \doteq ]r_{min}, r_{max}[ .$$

**Step 5.** We will now define a patchy control on a domain containing  $\Lambda \setminus \mathcal{E}$ . Given  $\eta > 0$  small, for each  $x \in \Lambda_i$  consider the point (see Figure 5)

$$p_i^x \doteq \frac{2}{3}x + \frac{1}{3}x_i + \eta \frac{x - x_i}{|x - x_i|}$$

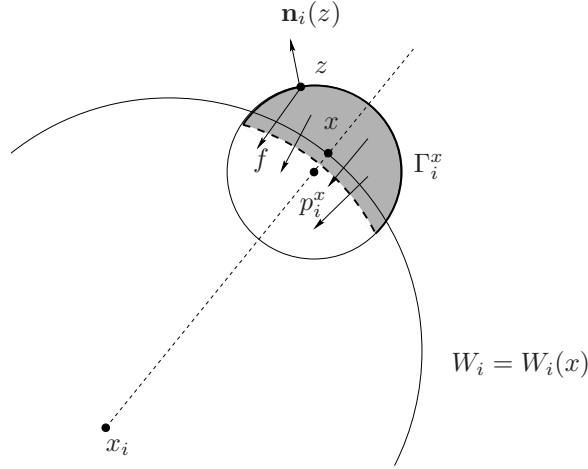


Figure 5: Construction of a lens-shaped patch.

and the ball  $B_i^x = B(p_i^x, |x - x_i|/3)$  centered at  $p_i^x$  with radius  $r = |x - x_i|/3$ . By (69), there exists a stabilizing control value  $u = u_i^x \in \mathbf{U}$  such that

$$\nabla W_i(x) \cdot f(x, u_i^x) \leq -h(x) + \varepsilon/4. \quad (76)$$

Consider the lens-shaped region

$$\Gamma_i^x \doteq B_i^x \setminus \overline{B}(x_i, |x - x_i| - \eta).$$

Its upper boundary will be denoted as

$$\partial^+ \Gamma_i^x \doteq \partial \Gamma_i^x \setminus \overline{B}(x_i, |x - x_i| - \eta).$$

Also, for  $z \in \partial^+ \Gamma_i^x$ , we write  $\mathbf{n}_i(z)$  for the outer unit normal at the point  $z$ . Our first goal is to show that, by choosing  $\eta > 0$  sufficiently small, the following holds:

$$\nabla W_i(z) \cdot f(z, u_i^x) \leq -h(z) + \varepsilon/2, \quad \forall z \in \Gamma_i^x, \quad (77)$$

$$\mathbf{n}_i(z) \cdot f(z, u_i^x) \leq -\eta, \quad \forall z \in \partial^+ \Gamma_i^x, \quad (78)$$

$$\text{diam } \Gamma_i^x \leq \lambda', \quad (79)$$

where  $\lambda' > 0$  is the constant fixed in Step 1 for property (iv). Moreover, the constant  $\eta > 0$  can be chosen uniformly valid for all  $i = 1, \dots, q$  and all  $x \in \Lambda_i$ .

For fixed  $i, x$  this is clear because, as  $\eta \rightarrow 0$ , the diameter of the set  $\Gamma_i^x$  approaches zero and therefore (79) is immediate. Moreover, as  $z$  varies on the upper boundary  $\partial^+ \Gamma_i^x$ , all the unit normals  $\mathbf{n}_i(z)$  approach the vector

$$\frac{x - x_i}{|x - x_i|} = \frac{\nabla W_i(x)}{|\nabla W_i(x)|}.$$

Therefore, both inequalities (77)–(78) follow from (76) and continuity of  $f, h, \nabla W_i$ .

We now observe that  $f = f(x, u)$  is uniformly continuous on the compact domain  $\overline{\Lambda} \times \mathbf{U}$  and  $h(x)$  is uniformly continuous on the compact domain  $\overline{\Lambda}$ . Moreover, on each set  $\Lambda_i$ , the gradient  $\nabla W_i(x)$  is uniformly Lipschitz continuous and bounded away from zero. Also, the radius of each

level set, where  $W_i$  is constant, by (74) is uniformly bounded above and below. This allows us to choose a constant  $\eta > 0$  uniformly valid for all  $i, x$ .

**Step 6.** Estimates (77)–(78) are exactly what we need on the region where  $W = W_i$ . However, they do not provide any insight about the behavior of  $f(\cdot, u_i^x)$  on the set

$$\Gamma_i^x \cap \{z \in \mathbb{R}^d ; W(z) = W_j(z) < W_i(z), j \neq i\} , \quad (80)$$

when this set is nonempty. In order to take care of this situation, we observe that the set where  $W_i = W_j$  is always a hyperplane, say

$$\mathcal{H}_{ij} \doteq \{x \in \mathbb{R}^d ; W_i(x) = W_j(x)\} = \{x \in \mathbb{R}^d ; \mathbf{n}_{ij} \cdot x = c_{ij}\} . \quad (81)$$

for a suitable constant  $c_{ij}$  and a unit normal vector  $\mathbf{n}_{ij}$ . The orientation of  $\mathbf{n}_{ij}$  will be chosen so that

$$\{x \in \mathbb{R}^d ; W_i(x) < W_j(x)\} = \{x \in \mathbb{R}^d ; \mathbf{n}_{ij} \cdot x < c_{ij}\} .$$

We now claim that, by choosing  $\eta > 0$  sufficiently small, uniformly w.r.t.  $i, x$ , only one of the following two cases occurs.

CASE 1: At every point  $z \in \Gamma_i^x \cap \mathcal{H}_{ij}$  one has

$$\mathbf{n}_{ij} \cdot f(z, u_i^x) < -\eta . \quad (82)$$

CASE 2: At every point  $z \in \Gamma_i^x$  such that  $W(z) = W_j(z)$  one has

$$\nabla W_j(z) \cdot f(z, u_i^x) \leq -h(z) + \varepsilon . \quad (83)$$

Indeed, assume that (82) fails. Then, there exists a point  $z^* \in \Gamma_i^x \cap \mathcal{H}_{ij}$  such that

$$\mathbf{n}_{ij} \cdot f(z^*, u_i^x) \geq -\eta .$$

By (81) and the orientation of the unit vector  $\mathbf{n}_{ij}$ , we can write

$$\nabla W_j(z^*) = \nabla W_i(z^*) - \beta \mathbf{n}_{ij}$$

for some constant  $\beta > 0$ , uniformly bounded above thanks to the bounds proved in Step 4. This now implies, relying on (77),

$$\begin{aligned} \nabla W_j(z^*) \cdot f(z^*, u_i^x) &= \nabla W_i(z^*) \cdot f(z^*, u_i^x) - \beta \mathbf{n}_{ij} \cdot f(z^*, u_i^x) \\ &\leq -h(z^*) + \varepsilon/2 + \beta \eta \leq -h(z^*) + 3\varepsilon/4 , \end{aligned} \quad (84)$$

provided that we choose  $\eta > 0$  sufficiently small. Since  $f$  and  $h$  are uniformly continuous and  $\nabla W_j$  is uniformly Lipschitz continuous, from (84) it follows that (83) is valid for all  $z$  sufficiently close to  $z^*$ . By reducing the size of  $\eta > 0$ , by Step 5 we can make the diameter of the lens-shaped domain  $\Gamma_i^x$  as small as we like and therefore, up to a last reduction of  $\eta$ , we finally get the inequality (83) for all  $z \in \Gamma_i^x$ .

**Step 7.** In order to define the patchy control, we now replace some of the domains  $\Gamma_i^x$  with smaller ones: we define the domain

$$\Omega_i^x \doteq \Gamma_i^x \setminus \bigcup_{j \in I_i} \{z \in \mathbb{R}^d ; W_j(z) \leq W_i(z)\} , \quad (85)$$

where  $I_i \subset \{1, \dots, q\}$  is the set of indices  $j \neq i$  for which CASE 1 applies. We recall that, if CASE 2 applies for the index  $j \neq i$ , this means that in points  $z \in \Gamma_i^x \cap \{\xi ; W_i(\xi) > W_j(\xi)\}$  one has both inequalities

$$\nabla W_i(z) \cdot f(z, u_i^x) \leq -h(z) + \varepsilon, \quad \nabla W_j(z) \cdot f(z, u_i^x) \leq -h(z) + \varepsilon.$$

In particular, the latter one ensures that the inequality (32) is satisfied in  $\Gamma_i^x$  with  $U(x) = u_i^x$  also in those points where  $W = W_j < W_i$ . On the other hand, if CASE 2 does not apply and (32) could fail, we must be in CASE 1 and, therefore, the vector field  $f(z, u_i^x)$  is pointing towards the interior of

$$\Gamma_i^x \cap \{\xi ; W_i(\xi) < W_j(\xi)\}$$

in all points  $z \in \mathcal{H}_{ij} = \{\xi ; W_i(\xi) = W_j(\xi)\}$ . Hence, we would like to remove from the domain  $\Gamma_i^x$  the region where (32) can fail. We just have to verify that  $f(\cdot, u_i^x)$  satisfies the inward-pointing conditions in all new points of the boundary, i.e. in points of  $\partial\Omega_i^x \cap \mathcal{H}_{ij}$ . For points  $z \in \Omega_i^x \cap \mathcal{H}_{ij} \setminus \partial\Gamma_i^x$ , inward-pointing condition is an immediate consequence of (82); for points  $z \in \Omega_i^x \cap \mathcal{H}_{ij} \cap \partial^+\Gamma_i^x$  the condition is satisfied because, for  $t > 0$  small enough, one has  $z + tf(z, u_i^x) \in \Omega_i^x$  and hence the vector is inside the Bouligand cone as requested by (11).

We also remark that, since  $\Omega_i^x \subseteq \Gamma_i^x$ , from (79) we get  $\text{diam } \Omega_i^x \leq \lambda'$  for all  $i, x$ .

The family of all domains  $\Omega_i^x$ , as  $i \in \{1, \dots, q\}$  and  $x$  ranges over the closure of the set  $\Lambda \setminus \mathcal{E}$ , represents our candidate for the covering in the definition of patchy feedback. Hence, we now select finitely many domains  $\Omega_i^x$  which cover the compact set  $\Lambda \setminus \mathcal{E}$ . This step, however, must be done with some care because on the lower portion of the boundary

$$\partial^-\Omega_i^x \doteq \partial\Omega_i^x \cap \overline{B}(x_i, |x - x_i| - \eta) .$$

the vector field  $f(\cdot, u_i^x)$  may not be inward-pointing. To cope with this problem, we first observe that there exists a uniform constant  $\sigma > 0$  such that

$$W_i(z) \leq W_i(x) - \sigma, \quad (86)$$

for every  $i, x$  and every  $z \in \partial^-\Omega_i^x$ .

We now set  $M^* \doteq \max_{\Lambda} W$ ,  $m^* \doteq \min_{\Lambda \setminus \mathcal{E}} W$  and  $N \doteq \lfloor \frac{M^* - m^*}{\sigma} \rfloor + 1$ , and split the domain  $\Lambda \setminus \mathcal{E}$  in sub-domains of the form

$$\mathcal{L}_\ell \doteq \{x \in \Lambda \setminus \mathcal{E} ; M^* - (\ell + 1)\sigma \leq W(x) < M^* - \ell\sigma\} .$$

For each  $\ell \in \{0, \dots, N - 1\}$ , we cover the compact set  $\overline{\mathcal{L}}_\ell$  with finitely many domains  $\Omega_i^x$ , with  $x \in \overline{\mathcal{L}}_\ell$ . After a relabelling of both the domains and the corresponding vector fields from (76), this yields the collection (see Figure 6)

$$(\Omega_{\ell, \alpha}, f(\cdot, U_{\ell, \alpha})) , \quad \alpha = 1, \dots, N_\ell, \quad (87)$$

where  $U_{\ell, \alpha} = u_i^x$  whenever  $\Omega_{\ell, \alpha} = \Omega_i^x$ , and the values  $u_i^x$  had been chosen in Step 5.

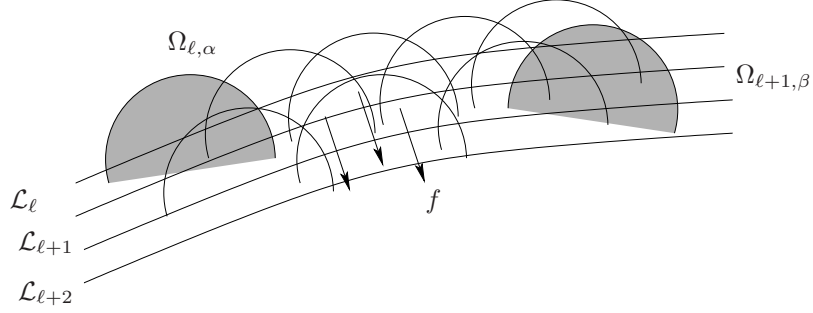
On this collection (87), we define the lexicographic order “ $\prec$ ” from (19) and we claim that the above construction yields a patchy vector field on  $\mathcal{D} \doteq \bigcup \Omega_{\ell, \alpha} \supseteq \Lambda \setminus \mathcal{E}$ :

$$f(x) \doteq f(x, U_{\ell, \alpha}) \quad \text{iff} \quad x \in \Omega_{\ell, \alpha} \setminus \bigcup_{(\ell, \alpha) \prec (\ell', \alpha')} \Omega_{\ell', \alpha'} . \quad (88)$$

Indeed, according to Remark 2.3, it suffices to check that, for each patch  $\Omega_{\ell, \alpha} = \Omega_i^x$ , the vector field  $f(\cdot, U_{\ell, \alpha}) = f(\cdot, u_i^x)$  is inward-pointing at every point of the set

$$(\partial\Omega_{\ell, \alpha} \cap \mathcal{D}) \setminus \bigcup_{(\ell, \alpha) \prec (\ell', \alpha')} \Omega_{\ell', \alpha'} .$$



Figure 6: The patches  $\Omega_{\ell, \alpha}$  used to cover  $\overline{\Lambda \setminus \mathcal{E}} = \bigcup \overline{\mathcal{L}_\ell}$ .

In the present case, this is clear, because we have already remarked that the vector field is inward-pointing on

$$\partial\Omega_i^x \cap (\partial^+\Gamma_i^x \cup \mathcal{H}_{ij}) .$$

As such, the only boundary points where  $f(\cdot, u_i^x)$  can fail to be inward-pointing are those on the lower boundary  $\partial^-\Omega_i^x$ . From  $x \in \overline{\mathcal{L}_\ell}$ , we deduce  $W(x) \leq M^* - \ell\sigma$ , and hence by (86)

$$W(z) \leq M^* - (\ell + 1)\sigma \quad \forall z \in \partial^-\Omega_i^x .$$

Therefore, given any point  $z \in \partial^-\Omega_i^x \cap \mathcal{D}$ ,  $z \in \overline{\mathcal{L}_{\ell'}}$  for some  $\ell' > \ell$ . Thus, either  $z \in \partial\mathcal{D}$  and there is nothing to verify, or  $z$  is contained in a patch  $\Omega_{\ell', \alpha'}$  with  $\ell' > \ell$ , as required in Remark 2.3.

**Step 8.** We want to show that, with the definition of the patchy feedback  $U$  given in Step 7, there holds

$$\nabla W(z) \cdot f(z, U(z)) \leq -h(z) + \varepsilon , \quad (89)$$

for all  $z \in \mathcal{D} \setminus \mathcal{E}$  in which  $\nabla W(z)$  is defined. But this is now immediate since, if  $(\ell, \alpha)$  is the patch index such that  $z \in \Omega_{\ell, \alpha}$  but  $z \notin \Omega_{\ell', \alpha'}$  for  $(\ell, \alpha) \prec (\ell', \alpha')$ , then there exist indices  $(i, x)$  such that  $\Omega_{\ell, \alpha} = \Omega_i^x$  and hence either  $W(z) = W_i(z)$  and the inequality follows from (77), or  $W(z) = W_j(z)$  where  $j$  is an index for which CASE 2 applies and therefore the inequality follows from (83).

To conclude the proof it remains to verify that (89) extends to (32) in points where  $\nabla W$  is not defined. First, observe that points  $\xi$  where  $W$  is not differentiable are characterized by the set

$$\mathcal{I}(\xi) \doteq \left\{ i \in \{1, \dots, q\} ; W(\xi) = W_i(\xi) \right\}$$

containing more than one element, and thus any singular point belongs to one or more hyperplanes  $\mathcal{H}_{ij}$  of the form (81), for  $i \neq j$  and  $i, j \in \mathcal{I}(\xi)$ . Now, fix  $x$  such that  $\mathcal{I}(x)$  contains multiple indices and assume  $U(x) = u_j^\xi$  for a suitable pair  $(\xi, j)$ , the control  $u_j^\xi \in \mathbf{U}$  being as in (76). Considering again that  $W$  is the minimum of the finite family of quadratic functions  $W_1, \dots, W_q$ , it is easy to verify that there exists  $\bar{\delta} > 0$  and  $\iota \in \mathcal{I}(x)$  such that for  $\delta \in ]0, \bar{\delta}[$

$$W(x + \delta f(x, u_j^\xi)) = W_\iota(x + \delta f(x, u_j^\xi))$$

and thus  $f(x, u_j^\xi) \bullet W(x) = f(x, u_j^\xi) \bullet W_\iota(x) = \nabla W_\iota(x) \cdot f(x, u_j^\xi)$ .

Now, if  $\iota$  is an index such that CASE 1 (82) happens, then we claim it must be  $\iota = j$  and thus (77) gives

$$f(x, u_j^\xi) \bullet W(x) = \nabla W_j(x) \cdot f(x, u_j^\xi) \leq -h(x) + \varepsilon/2 .$$

Indeed, if it were  $\iota \neq j$ , we would have  $x \in \partial\Omega_j^\xi \cap \mathcal{H}_{j\iota}$  and the strict inward pointing condition (82) would imply that for  $\delta$  small enough

$$W(x + \delta f(x, u_j^\xi)) = W_j(x + \delta f(x, u_j^\xi)) < W_\iota(x + \delta f(x, u_j^\xi)),$$

which in turn contradicts the choice of  $\iota$ .

Finally, if  $\iota$  is an index such that CASE 2 (83) happens, then  $x \in \Omega_j^\xi \cap \mathcal{H}_{j\iota}$  and (83) itself ensures

$$f(x, u_j^\xi) \bullet W(x) = \nabla W_\iota(x) \cdot f(x, u_j^\xi) \leq -h(x) + \varepsilon.$$

**Step 9.** We now want to prove that it is possible to find  $\sigma > 0$  and a subcollection of open domains extracted from  $\{\Omega_{\ell,\alpha}\}_{(\ell,\alpha) \in \mathcal{A}}$  with the properties required in Remark 3.2. By taking  $\sigma > 0$  to be the constant in (86), the definition of the “slices”  $\mathcal{L}_m$  in the remark coincides with the one given in Step 7, for  $m = 0, \dots, N-1$ . Fix a compact set  $K \subseteq \Lambda \setminus \mathcal{E}$  and assume that the domains used for the patches of  $U$  satisfy  $\text{diam } \Omega_{\ell,\alpha} \leq \lambda'$  for all  $(\ell, \alpha)$ . Then, we set for any  $m = 0, \dots, N-1$

$$\begin{aligned} \mathcal{I}_m &\doteq \{\beta = (m, \alpha) \in \mathcal{A} ; \Omega_{m,\alpha} \cap K \neq \emptyset\}, \\ \beta_m &\doteq \min_{\mathcal{I}_m} \beta, \quad \mathcal{B} \doteq \bigcup_{m=0}^{N-1} \mathcal{I}_m. \end{aligned}$$

We know that each  $\mathcal{I}_m$  has finite cardinality because the lens-shaped domains  $\Omega_{m,\alpha}$  were a finite covering of  $\overline{\mathcal{L}}_m$  (see again Step 7 above). Hence, it follows immediately that the indices  $\beta_m$  are well defined, that they increase when  $m$  increases and that

$$K \cap \overline{\mathcal{L}}_m \subseteq \bigcup_{\substack{\beta \in \mathcal{B} \\ \beta_m \leq \beta < \beta_{m+1}}} \Omega_\beta, \quad \forall m \in \{0, \dots, N-1\}.$$

Also, it is easy to verify that (i) holds, given that the sets  $\{\Omega_{\ell,\alpha}\}$  formed a covering of  $\Lambda \setminus \mathcal{E}$  and that  $\text{diam } \Omega_{\ell,\alpha} \leq \lambda'$ . Hence, it remains to show that (40) is verified. But we have already proved in Step 7 that, on each  $\Omega_{m,\alpha} = \Omega_i^x$ ,  $f(\cdot, u_i^x)$  can violate the inward-pointing condition only in points of  $\partial^- \Omega_i^x$  and that these points belong to  $\bigcup_{m' > m} \mathcal{L}_{m'}$ . Therefore, (40) holds and the proof is complete.  $\diamond$

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